UNIVERSITY PHYSICS

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UNIVERSITY PHYSICS

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PART ONE

GENERAL PHYSICS

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PREFACE

This book is primarily intended for students taking a First and Second Year Course in Physics at a University. It is designed for preparation for examinations of the standard of Part I of the Natural Sciences Tripos at Cambridge, the B.Sc. General Degree of London and, by the omission of those sections marked with an asterisk, for Intermediate students who have already studied the elements of Physics at school or elsewhere.

It must be remembered that at the stage covered by this book, students will not yet have become specialists in Physics. The writer has had frequent experience of students who, during a period when they are studying two or three additional subjects, feel a great need for **one book** on Physics which contains the basic information which they must acquire. It is not suggested that this book has made others unnecessary or, more particularly, that it has rendered lectures superfluous. It remains as important as ever for students to read widely and to acquire experience of the different methods of treatment of a subject which only a diversity of Lecturers can supply.

Finally, it is becoming more and more recognized, at least as an ideal, that material usually given in formal lectures can be quite as well acquired from good text-books and that lectures will gradually develop into a tuitional tutorial system under which the time and energy of the lecturer can be devoted to the detailed elucidation of difficult points, apt illustrations and demonstrations, the discussion of essays and exercises done by the student, and the exercise of personality to engender an enthusiasm without which a subject remains "dry bones".

F. C. CHAMPION.

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PART I

GENERAL PHYSICS

Aim of Physical Science

CHAPTER I

Introduction

1. The Aim of Physical Science.

The aim of physical science is the reference of natural events, which at first may seem quite disconnected, to certain general laws, the number of which should be as few as possible. For example, the vast majority of objects, when unsupported from beneath, fall to the ground with an acceleration independent of the particular object considered. This behaviour is evidence for a **general law**, in this case the law of universal gravitation. Again, bodies charged with electricity are found to exert forces on each other and experiment shows that these electrical forces are likewise governed by a general law. As further and different physical phenomena, such as light and sound, are revealed by our senses, so general laws are deduced governing each of these physical phenomena. Diligent inquiry results in the number of laws deduced in this way rapidly becoming inconveniently large for the human mind, particularly for the human memory, and attempts are therefore made to relate these different laws and reduce them all to fewer and still more basic laws. For example, light may now be explained almost entirely in terms of electricity, and sound in terms of the general mechanical laws of motion. It is broadly true to say that the aim of physical science is the reduction of all physical matter to a single primary "stuff". A knowledge of the different forms that this "stuff" can then take, together with the mutual forces exerted by the "bits" of "stuff" would, it is hoped, account for all known phenomena. We should hasten to inform the student that we are still far short of our aim, which, indeed, must always probably remain as an ideal to future generations of physicists. It is clear, also, that before suggestions of any value can be put forward, a detailed knowledge of a great field of physical phenomena must be possessed by the natural philosopher. This book contains an account of some of the more elementary of these phenomena.

2. The Concepts of Length and Time.

The concept of **length** or **extension** arises from at least two of the basic human sensory perceptions, those of sight and touch. Visually, certain discontinuities are observed, marking off one object from another. The human eye perceives that these discontinuities are not

all the same size, and so we obtain the notion of one body having greater extension than another. Confirmation of the relative sizes is provided over a limited range by touch, some objects being less than, say, the span of outstretched arms and others greater.

The concept of time originates in various ways, the strongest evidence being provided by the regular alternation of natural phenomena such as night and day and the seasons.

We defer the discussion of the choice of definite units of length and time until Chap. V, since we wish to use the results of earlier chapters in providing the student who has already passed the primary stage in Physics with more precise notions than can be gained without some quantitative background.

3. States of Matter.

The concept of **matter** arises exclusively from the sensation of touch. The student will be familiar with optical illusions in which various objects *appear* to be present as in the celebrated "Pepper's ghost" (see Part III). Whether they are "really" there depends upon whether they are "as sensible to feeling as to sight". Common experience also informs us that there are three well-defined states of matter—solids, liquids and gases. The distinction between the three states is much clarified by the **molecular theory of matter.** The idea that matter is not continuous but consists eventually of small, separate, indivisible parts or molecules, received its first firm experimental support from chemistry, in the shape of Dalton's laws of the combining powers of the elements. Since that time a multitude of evidence has arisen, chiefly connected with the kinetic theory of matter discussed in Part II, to support the molecular hypothesis. Briefly, in the **solid state** the molecules are considered to have no translational motion, but to be capable of a certain amount of rotational, and particularly vibrational, energy. That is, they are considered mainly as vibrating within very narrow limits about fixed centres. The amplitude of these vibrations is much too small to be detected even with a microscope, and consequently a solid maintains its fixed shape over an indefinite period of time. Occasionally one of the molecules will acquire sufficient vibrational energy to break away completely from the rest of the solid. Such wandering molecules give rise to the **diffusion of solids** described on p. 129. The vapour pressure of a solid also arises in this way. In most solids it is very small; for example, gem-stones maintain their shape and weight over centuries. On the other hand, a block of solid camphor decreases in size appreciably in a few months.

In the **liquid state**, one of the chief characteristics of which is that a liquid takes up the shape of the vessel in which it is placed, the molecules have a certain amount of translational, rotational and

vibrational energies. How the energy is divided between the three forms depends very much on the nature and temperature of the liquid. In general, however, the vapour pressure of liquids is much higher than that of solids and the rate of diffusion much larger, so molecules of liquids possess a considerable translational energy.

In the **gaseous state**, all the molecules possess a high translational energy although they generally possess vibrational and rotational energy as well. Their high translational energy explains the ability of a gas to " fill " any vessel into which it is placed, the existence of gas pressure, and the high rate of gaseous diffusion.

In the solid state the density is high and the molecules are comparatively tightly packed, the average distance between the centres of two adjacent molecules being about 10^{-8} cm. With gases, the density is low and the average distance between the molecules comparatively large, being about 10^{-5} cm., although it depends, of course, on the gas pressure. Liquids are in an intermediate state. When a solid is heated we imagine on the molecular picture that the molecules vibrate with larger and larger amplitude as the temperature is raised until they break away from their fixed centres. The solid body then loses its definite shape, is said to melt, and has been transformed into the liquid state. On continuing the heating process the translational energy of the molecules increases until, when the liquid boils, the molecules have sufficient energy to escape completely from the main bulk of the liquid and the liquid is being transformed into the gaseous state. In the solid and liquid states the molecules are held together by forces of **mutual attraction** which are, incidentally, electrical and not gravitational in nature. In the gaseous state, however, the molecules are sufficiently far apart to be independent of each other unless the pressure and therefore the density of the gas is high. These facts have made the development of a **kinetic theory of gases** comparatively simple, but owing to our ignorance of the precise nature of intermolecular forces in solids and liquids, these have not proved so amenable to treatment. The molecular hypothesis is of the utmost importance in obtaining an insight into physical processes and the student should always attempt to form a molecular picture of the different physical phenomena as he comes in contact with them.

It is true that in recent years the molecule and atom have been " split ", so that they can no longer be regarded as the indivisible units they were originally thought to be. For the vast majority of physical phenomena, however, matter behaves as though the molecules were the ultimate units of matter, and in the broad description of physical events a more detailed examination is unnecessary and undesirable. The simplest explanation, provided it explains all the facts, should be the aim of physical theory, not the clever elaboration of intricate hypotheses, however ingenious.

CHAPTER II

Kinematics

1. Kinematics.

Kinematics is the study of the space-time variations of bodies relative to a certain set of co-ordinates. We find it convenient to commence with the definition of **velocity.**

2. Translational Velocity.

Translational velocity is defined as the rate of change of position of a body.

Thus, if we consider the body as moving along the x-axis of our co-ordinate system, its **average velocity** over any short interval of time Δt is $\Delta x/\Delta t$, where Δx is the short distance traversed in the short interval Δt. Proceeding to the limit according to the principles of the differential calculus, we define the **instantaneous velocity** by dx/dt, the derivative, or differential coefficient, of x with regard to t, which can be calculated when x is known as a function of t.

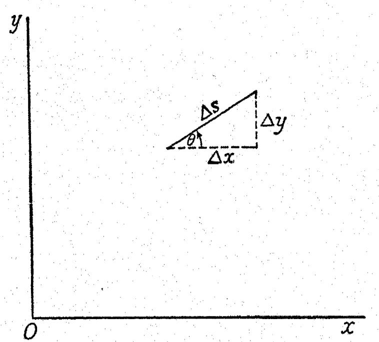

Fig. 1

In the simple example considered we have confined the motion to one particular direction. If the motion of the body was at an angle to the x-axis but was confined to the xy-plane, the actual velocity v_s at any instant would be ds/dt, where s is the distance, measured along the path from some fixed point on it up to the point reached at time t; so that s is a function of t.

Referring to fig. 1, if the short length Δs is inclined to the x-axis at an angle θ, the distance traversed in the x-direction is

$$\Delta x = \Delta s \,.\, \cos\theta.$$

Similarly, in a direction parallel to the y-axis, the distance traversed is

$$\Delta y = \Delta s \,.\, \sin\theta.$$

(F 729)

Now the velocities in these two directions are by definition $v_x = dx/dt$ and $v_y = dy/dt$; hence

$$v_x = dx/dt = (dx/ds)\,(ds/dt) = \cos\theta\,.\,ds/dt = v_s\cos\theta, \quad (2.1)$$

and
$$v_y = dy/dt = (dy/ds)\,(ds/dt) = \sin\theta\,.\,ds/dt = v_s\sin\theta; \quad (2.2)$$

also from (2.1) and (2.2), by squaring and adding,

$$v_s^2 = v_x^2 + v_y^2. \quad (2.3)$$

If the motion of the body is not confined to a plane, the short distance Δs, as shown in fig. 2, can still be resolved into two distances,

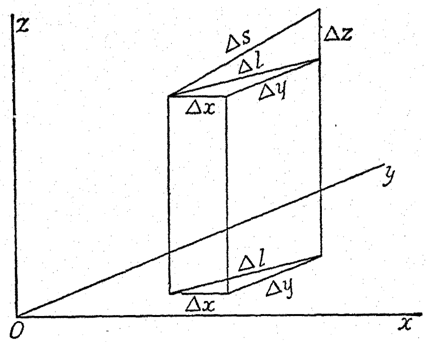

Fig. 2

one Δz parallel to the z-axis and the second, Δl, perpendicular to Δz and therefore parallel to the xy-plane. We note that Δl has itself **components** Δx and Δy, and since

$$\Delta l^2 = \Delta x^2 + \Delta y^2$$

and
$$\Delta s^2 = \Delta z^2 + \Delta l^2,$$

therefore
$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2. \quad (2.4)$$

Considering equations (2.1), (2.2) and (2.3), we see that velocities, like distances, can be *resolved* in any direction. Hence, extending (2.4) to the case of velocities,

$$v_s^2 = v_x^2 + v_y^2 + v_z^2. \quad (2.5)$$

Conversely, if velocities v_x, v_y and v_z, parallel to the three axes respectively, are impressed on or possessed by a body, the latter is said to acquire or to have a **resultant velocity** v_s, given by equation (2.5), having **components** v_x, v_y and v_z.

3. Angular Velocity.

If instead of using Cartesian co-ordinates to define the position of the body, as in § 2, we use angular co-ordinates, and if we confine our discussion for simplicity to planar motion, the position of the body will be defined at any instant by the co-ordinates (r, θ) with respect to some fixed initial direction. If θ changes by an infinitely small amount $d\theta$ in an infinitely short interval dt, the **angular velocity is given by $d\theta/dt$** and is usually represented by $\boldsymbol{\omega}$.

4. Vector and Scalar Quantities.

In § 2 we treated resultant velocity by *algebraic and trigonometrical analysis*. An alternative method is by *graphical treatment* according to the rules of *vector analysis*.

We define a scalar quantity as one possessing magnitude but not direction.

Examples of scalars are volume (cubic centimetres) and quantity of heat (calories). We may represent a scalar quantity graphically as a line, the length of which is proportional to the magnitude of the quantity involved.

There is another class of entities which possess direction as well as magnitude, and are called vectors.

Examples of vectors are velocity (cm./sec.) and magnetic field strength (oersteds). As with scalars we can again represent the magnitude by a line of certain length, but in addition we must mark the *direction* of the quantity with respect to some definite co-ordinate system.

The essential difference between scalar and vector quantities is made manifest if we wish to add two or more of these quantities. Thus the addition of 20 c.c. to 30 c.c. is (theoretically) 50 c.c. or the addition is *simply arithmetical*. On the other hand, the resultant velocity acquired by a body on which is impressed successively velocities of 10 m.p.h. and 20 m.p.h. will not be 30 m.p.h. except for the very special case where the two velocities are in the same direction.

To combine two or more velocities in the general case we may proceed as in § 2, resolving each velocity into components along a given set of co-ordinate axes, add the corresponding components algebraically, and then determine the resultant according to equation 2.5.

A graphical method, which is sometimes simpler, is however available. Represent the first velocity by a vector AB, that is, draw a line AB in the direction of the first velocity and of length proportional to its magnitude. Then from the point A draw a second vector AD to represent the second velocity. Complete the parallelogram ABCD, and the diagonal AC is the resultant velocity. This construction is known as the **Parallelogram of Velocities.** It is clearly unnecessary to complete the entire parallelogram, one half or a **Triangle of Velocities**

being sufficient. Since the third side of a triangle is given by well-known trigonometrical formulæ, in practice the resultant is often *calculated*, for example, by the *cosine formula.*

Example. Velocities of 20 m.p.h. and 30 m.p.h. are impressed on a body in directions due E. and N.E. respectively. Find the resultant velocity acquired by the body.

$$v^2 = 20^2 + 30^2 + 2 . 20 . 30 . \cos 45°; \; v = 46{\cdot}4 \text{ m.p.h.}$$

The direction of the resultant velocity is easily found graphically; alternatively it is found from the sine formula,

$$30/\sin\theta = 46{\cdot}4/\sin 135°; \; \theta = 27{\cdot}3° \text{ N. of E.}$$

5. Acceleration.

Acceleration is defined as the rate of change of velocity.

By complete analogy with the preceding considerations on velocity, for a body moving with instantaneous velocity v along the x-axis, the **linear acceleration** will be

$$a = dv/dt = (d/dt)(dx/dt) = d^2x/dt^2, \quad . \quad . \quad . \quad (2.6)$$

since $v = dx/dt.$

For a body moving in a plane at an angle θ to the x-axis, we have by similar reasoning

$$a_x = dv_x/dt = (d/dt)(v \cos\theta) = \cos\theta \, . \, dv/dt = a \cos\theta \qquad (2.7)$$

and $$a_y = dv_y/dt = (d/dt)(v \sin\theta) = \sin\theta \, . \, dv/dt = a \sin\theta, \qquad (2.8)$$

where a_x and a_y are the components of the acceleration along the x- and y-axis respectively.

For a body moving in *any* direction,

$$a_s^2 = a_x^2 + a_y^2 + a_z^2, \quad . \quad . \quad . \quad . \quad . \quad (2.9)$$

where a_x, a_y and a_z are the component accelerations.

If the angular velocity of the body is changing, the angular acceleration is defined as the rate of change of angular velocity.

That is $$\alpha = d\omega/dt = d^2\theta/dt^2. \quad . \quad . \quad . \quad . \quad (2.10)$$

It is clear that, like velocity, *acceleration is a vector quantity* and will follow the rules for vector addition. **Resultant accelerations** may therefore be obtained by graphical construction, the diagram being referred to as the **Triangle** or **Parallelogram of Accelerations.**

6. Relative Motion.

If we consider two bodies A and B, moving with velocities u and v, with respect to a fixed co-ordinate system, along the common x-axis,

it is of importance to inquire with what velocity either body appears to be moving to an observer situated on the other, that is, the **relative velocity** of say B to A. In this simple case, direct experiment with a stop-clock and measuring rod shows that the relative velocity is $v \pm u$. In particular, if the two velocities are in the *same* direction it is found that the negative sign gives the relative velocity.

The relative velocity of B to A is therefore obtained by reversing the velocity of A and adding this to the velocity of B.

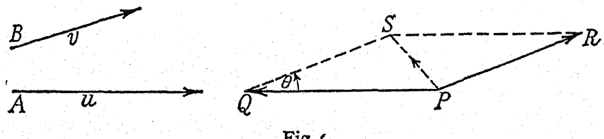

Fig. 3

If, however, the two velocities are inclined to each other we must combine the velocities according to the rules of vector addition. Graphically the construction is as follows. Draw a line PQ parallel to the velocity u and of length proportional to its magnitude; PQ is, however, reversed in direction. Draw PR parallel to the velocity v and of length proportional to its magnitude. Then the relative velocity of B to A is obtained by completing the parallelogram and is given in magnitude and direction by the diagonal PS (fig. 3).

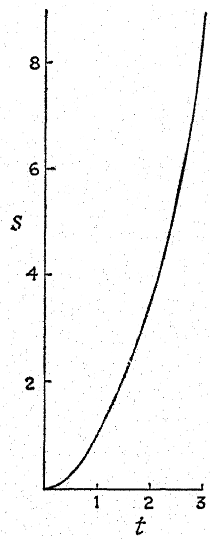

Fig. 4

We may find the magnitude of PS analytically. Take the origin of co-ordinates at some point on the body A and the x-axis along the direction of motion of A. Then the components of the velocity of B along and perpendicular to this axis are $v \cos\theta$, $v \sin\theta$ respectively; the relative velocity of B to A will be obtained by compounding $v \sin\theta$ and $(v \cos\theta - u)$ at right angles. Hence

$$\begin{aligned} v_{\text{rel}}^2 &= v^2 \sin^2\theta + v^2 \cos^2\theta - 2vu \cos\theta + u^2 \\ &= u^2 + v^2 - 2uv \cos\theta \\ &= \text{PS}^2, \end{aligned}$$

as we see from fig. 3.

7. Space-Time Relations.

The graph obtained by plotting the position of a body against the time is referred to as the **space-time curve.** The space-time curve for a body starting from rest and moving with uniform acceleration is shown in fig. 4, which is the graphical expression of equation (2.15) for the special case where

$x_0 = v_0 = 0$, the space s in this case being equal to x. Since acceleration is defined as the derivative of velocity with respect to time, the velocity will be given by the integral of the acceleration with respect to time. Thus, since from (2.6)

$$a = dv/dt,$$

if the acceleration is constant, integration gives

$$at = v + \text{constant of integration.} \quad . \quad . \quad . \quad . \quad . \quad (2.11)$$

To determine the value of the constant of integration, we must substitute in (2.11) some known value of v at some known time t. Before the body commences to move, that is at time $t = 0$, $v = 0$; hence from (2.11), the constant of integration is zero and

$$v = at. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.12)$$

If, however, the body had an initial velocity of v_0 parallel to the x-axis, $v = v_0$ when $t = 0$ and substitution in (2.11) shows that the constant of integration is $(-v_0)$. Hence, generally, the relation between velocity, time and constant acceleration is

$$\boldsymbol{v = v_0 + at}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.13)$$

To obtain the relation between distance and time we use (2.13), remembering that $v = dx/dt$; hence

$$dx = v_0 \,.\, dt + at \,.\, dt.$$

Integrating, we obtain

$$x = v_0 t + \tfrac{1}{2}at^2 + \text{constant of integration.} \quad . \quad (2.14)$$

The constant of integration is obtained as before by considering the value of x at the time $t = 0$. If the body is situated at the origin of co-ordinates at the time $t = 0$, then $x = 0$, and substitution in (2.14) gives constant of integration $= 0$. More generally, if the body is at some point $x = x_0$ when $t = 0$, substitution in (2.14) shows that the constant of integration is equal to x_0. The general form of (2.14) is therefore

$$\boldsymbol{(x - x_0) = v_0 t + \tfrac{1}{2}at^2}. \quad . \quad . \quad . \quad . \quad (2.15)$$

If we eliminate t from equations (2.13) and (2.15) we get

$$\boldsymbol{v^2 = v_0^2 + 2a(x - x_0)}, \quad . \quad . \quad . \quad . \quad (2.16)$$

a useful equation connecting velocity and acceleration with the distance traversed.

By similar reasoning we may deduce that the angular velocity of

a body at time t is connected with its angular velocity ω_0 at time $t = 0$, and the constant angular acceleration α, by

$$\omega = \omega_0 + \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.17)$$

Similarly, the angular displacement θ at time t is related to θ_0, its value at time $t = 0$, by

$$(\theta - \theta_0) = \omega_0 t + \tfrac{1}{2}\alpha t^2, \quad . \quad . \quad . \quad . \qquad (2.18)$$

by analogy with (2.15).

The preceding examples have referred to the simple case of motion with constant acceleration. In the more elementary considerations of physics there is only one important case with variable acceleration, namely, simple harmonic motion. This is discussed in detail in § 9, Chap. III.

8. Kepler's Laws of Planetary Motion.

The knowledge of physical science which is obtained by the study of kinematics is useful but very limited. One very important appli-

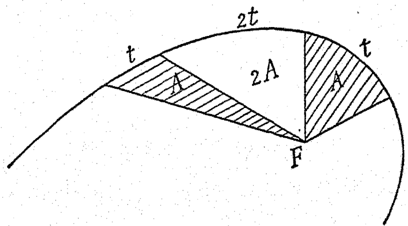

Fig. 5

cation is the description of the motion of the heavenly bodies. In particular, from the observations of Tycho Brahe, Kepler was able to formulate his three laws of planetary motion. These are:

1. *The planets describe ellipses round the sun, with the sun at one focus.*

2. *Equal focal areas are swept out in equal times* (see fig. 5).

3. *The squares of the times of revolution of the planets are proportional to the cubes of the major axes of their elliptical orbits.*

We shall refer briefly to the subject of planetary motion in § 7, Chap. VI, when discussing Newton's work on gravitation.

EXERCISES

1. Distinguish between vector and scalar quantities, giving examples from different branches of physics.

2. If the acceleration or retardation of a trolley-bus may not exceed 4 ft./sec.^{2} and its maximum allowable speed may not be greater than 30 m.p.h., find the shortest possible time for the bus to travel from rest to rest between two stations one mile apart. [131 sec.]

3. The retardation acting on a glider moving horizontally is proportional to the square of its velocity. Obtain the equations for the velocity-time and space-time curves. [$v = v_0/(kv_0t + 1)$, $ks = \log(kv_0t + 1)$.]

4. A steamer is travelling due N. at 15 knots and a vane on the mast-head points N.N.W.; the steamer turns E., whereupon the vane points N.N.E. Find the velocity and direction of the wind. [15 knots from N.W.]

5. An observer travelling due E. at a speed of 60 m.p.h. notices an object apparently moving due N. with an apparent speed of 30 m.p.h. Determine the true velocity and direction of motion of the object. [67·1 m.p.h. at $26\frac{1}{2}°$ N. of E.]

CHAPTER III

Dynamics of a Particle

In the subject of kinematics we were concerned with a simple description of *how* bodies move. Immediately we wish to discover *why* bodies move as they do, we enter the realm of **dynamics.** It was Newton who formulated the essential laws of this subject. We shall commence by stating his Three Laws of Motion, explaining later their precise significance.

1. Newton's Laws of Motion.

1. All bodies continue in their state of rest or of uniform motion in a straight line unless they are compelled to change that state by external forces.

2. Rate of change of momentum is proportional to the impressed force and takes place in the direction in which that force is acting.

3. Action and reaction are equal and opposite.

On examining the first law we see that any movement of a body from rest or any departure from uniform rectilinear motion is ascribed to an external agency—a **force.** The notion of force comes from a fundamental human sensory perception based on muscular effort. We note that it requires a definite muscular effort to stretch a spiral spring by a given amount. We observe further, by experiment, that, other things being equal, it always requires the same force to stretch the spring by the same amount.

Consider now the simple experiment in which the spring is stretched horizontally close to a large horizontal sheet of ice (frictionless plane), and suppose that two bodies A and B, lying on the ice, are successively attached to the spring. It is then found that if the spring is stretched initially by an equal amount in both cases, A and B commence to move, when released, with different horizontal accelerations. **The ratio of the accelerations acquired by two bodies when subjected to the same force is said to be inversely proportional to the masses of the bodies.** If we symbolize the force by P and the two

accelerations by a_1 and a_2, then, with proper choice of units, we may write

$$\left.\begin{aligned} a_1 &= \frac{P}{m_1} \\ a_2 &= \frac{P}{m_2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

or

$$P = m_1 a_1 = m_2 a_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

Since, by (3.2), a force can be measured by the acceleration it produces in a certain mass, it follows that **force, like acceleration, is a vector magnitude.** Also, it is implied by (3.2) that the **component force** in any direction is equal to the mass multiplied by the **component acceleration in that direction.**

2. Mass and Weight.

Equation (3.1) may be used to define the mass of a body.

The mass of a body is defined as a quantity associated with a body which is inversely proportional to its linear acceleration when a given force is applied to it. This definition is much to be preferred to the rather vague alternative sometimes quoted: "mass is the quantity of matter in a body".

In the above experiment with the spring we eliminated vertical forces by having the bodies moving horizontally on an almost frictionless plane. Actually, if any body situated near the surface of the earth is not supported, it is found to fall vertically with uniform acceleration, commonly denoted by g. This acceleration has a value of about 32·2 ft./sec.^{2} or 981 cm./sec.^{2} at sea-level near London. Hence, any body of mass m behaves, according to equation (3.1), as if it experienced a continual vertical force given by

$$\mathbf{P} = mg. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

This quantity mg is termed the **weight** of the body, that is, the weight of a body is defined as its mass multiplied by the acceleration due to gravity. Since g varies over the earth's surface (see § 8, Chap. VI), the weight of a body is variable, in contrast to its mass, which is constant. If gravity could be abolished the body would cease to have weight, but its mass would still exist and could be determined by experiments with a spiral spring or some similar arrangement.

3. Momentum.

Since acceleration is defined as rate of change of velocity, (3.2) may be written

$$\mathbf{P} = ma = m\frac{dv}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

or, since the mass is constant,

$$\mathbf{P} = \frac{d}{dt}(mv). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

The product of the mass and the velocity of a body is termed its momentum. Equation (3.5) is the direct mathematical expression of Newton's Second Law of Motion; the R.H.S. is the rate of change of momentum, while the L.H.S. is the force causing this change.

If we attempt to deflect a moving body by a force applied more or less at right angles to the direction of motion of the body, we find that deflection is difficult if the body is moving fast or if it has a large mass. It is the **product, mass × velocity,** that is, the **momentum** of a body, which determines its power of continuing comparatively unaffected by small disturbing forces.

4. Equilibrium.

An object, say a jar, resting on a shelf, is experiencing a continual force downwards due to the action of gravity, equal in magnitude to the weight of the jar. It is clear that the jar would move downwards under the action of this force unless there were an equal and opposite force or **reaction** acting on the jar. This reaction is provided by the shelf, which exerts a thrust upwards equal in magnitude to the weight of the jar downwards, and so **maintains the equilibrium.** While this equality of action and reaction is readily appreciated when bodies are in equilibrium, it must be studied in more detail if motion occurs. Thus if a horse pulls a cart with a force P, since action and reaction are equal and opposite the cart must be exerting, through the traces, a reaction P against the horse. Why then do both horse and cart move forward? The answer is that the forces are transmitted through the bodies concerned, and the ultimate cause of motion is the friction between the hoofs of the horse and the ground. Actually the reaction between the hoofs and the ground thrusting the horse forwards is accompanied by an equal and opposite action thrusting the earth backwards. Since the mass of the earth is so large, the effect on the earth is negligible, but the existence of such reactions is easily demonstrated by some simple experiment such as that illustrated in fig. 1.

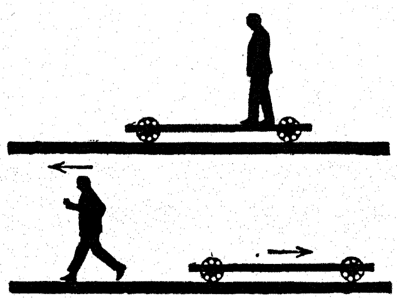

Fig. 1

As the man walks *forward* on the light truck the reaction of the man's feet on the truck causes the latter to move backwards, the

magnitude of the movement being governed by the relative masses of the man and the truck. Another example is the revolving cage in which white mice are sometimes housed. The mouse stands in a hollow cylinder mounted on a horizontal axis; when the mouse attempts to run forward it exerts a reaction on the cylinder and the latter revolves.

Returning to the example of the jar resting on the shelf, if we exert a horizontal force on the jar, experiment shows that the jar may move horizontally or it may commence to tilt according as the force is applied to the bottom or to the top of the jar. If the jar is of very small height, no tilting is observed. Further, if the force is fairly small, and the jar is made of some material like plasticine, instead of moving as a whole, it may undergo deformation.

It has been found convenient to introduce the following abstractions;

(1) **Material Particle.**

For many purposes we may consider the mass of a body to be concentrated at a certain point in the body. This point is termed the **centre of gravity.** Forces applied to the body which pass through this point cause the body to behave as though its whole mass were concentrated in a **material particle** placed at this point.

(2) **Rigid Body.**

If forces acting on a body do not pass through the centre of gravity then rotation is produced as well as translation. Deformation of a body due to forces requires special study (see Chap. VIII on *Elasticity*), and we therefore postulate that when considering rotation and translation we may assume initially that no deformations occur. We then define the body as a **rigid body.**

In practice there is no such thing as a material particle nor a rigid body. When a given force is applied to a body, some rotation, some translation and some deformation occur simultaneously. However, if the body is hard and the force is moderate, the deformation may be neglected compared with the rotation and translation. Finally, if the resultant force passes, as far as is experimentally possible, through the centre of gravity, rotation is absent and the body moves with translational motion given by calculation based on the assumption that the body may be replaced by a material particle of the same mass situated at the centre of gravity.

We shall discuss the subjects of centre of gravity and equilibrium more fully in the chapter on *Statics*.

5. Work.

A force is said to do work when its point of application moves, and the measure of the work done is defined as the product of the force and the distance moved in the direction of the force.

If the force is denoted by P and the distance by s, the work done, W, is given by

$$\mathbf{W} = \mathbf{P}s. \quad \quad (3.6)$$

In fig. 2, the work done when the point of application of the force moves from A to B is given by $Pl \cos\theta$. If θ is zero, the work done is Ps; on the other hand, if $\theta = 90°$, $\cos\theta = 0$, and therefore no work is done by a force when movement takes place at right angles to the force.

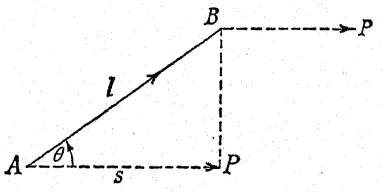

Fig. 2

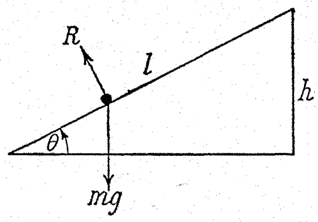

Fig. 3

Since the force on a body due to gravity is equal to its weight mg, the **work done against gravity** when a weight mg is raised, or rises, a vertical distance h, is

$$\mathbf{W} = mgh. \quad \quad (3.7)$$

Further, if a weight is pushed a distance l up a frictionless plane inclined at θ to the horizontal as in fig. 3, the work done is

$$\mathbf{W} = mgl \sin\theta = mgh, \quad \quad (3.8)$$

where h is the vertical distance. This is because the weight mg acts vertically downwards and our definition of work done is force $\times$ component distance moved *in direction of the force.*

Equation (3.8) shows us that the weight behaves as though it had a component $mg \sin\theta$ down the plane, for the product of this effective weight $mg \sin\theta$ and the distance l moved in the direction of the component also gives us the work done. We may also note that there must be a **reaction** R, perpendicular to the plane and given by

$$R = mg \cos\theta, \quad \quad (3.9)$$

if there is equilibrium, for the component of the weight in a direction perpendicular to the plane is $mg \cos\theta$. This reaction R does no work when the weight is moved along the plane, since it acts at right angles to the direction of movement.

6. Energy.

The energy of a body is defined as its capacity for doing work. Consider a stone of mass m thrown vertically upwards with initial velocity v_0. It is acted upon by a uniform retarding acceleration or **retardation** g, and therefore, by (2.16), will attain a height h given by

$$0 = v_0^2 + 2(-g)h, \quad \ldots \ldots \quad (3.10)$$

since the velocity v at the highest point must be zero. From (3.10),

$$h = \frac{v_0^2}{2g}. \quad \ldots \ldots \ldots \quad (3.11)$$

Now from (3.7), work done against gravity is

$$W = mgh = \frac{mgv_0^2}{2g} = \tfrac{1}{2}mv_0^2 \quad \ldots \ldots \quad (3.12)$$

by (3.11).

The available work which may be obtained from a body of mass m possessing a velocity v_0 is therefore $\frac{1}{2}mv_0^2$, and this is, by definition, the energy of the body. We shall see later that energy may exist in several different forms. The form just considered is **energy possessed by the body in virtue of its motion,** and is termed **kinetic energy.**

Now at the top of its flight the velocity of the stone is zero. Its kinetic energy is therefore zero; yet it can regain that kinetic energy if it is allowed to fall back to the point from which it was projected. This hidden or *latent energy* which the stone possesses owing to its height above the earth's surface is termed **potential energy.** It is energy in virtue of the **position** of the body, and is capable of being converted into kinetic energy. This fact leads to the extremely important conclusion that the kinetic energy of the stone was *not destroyed* as its velocity fell to zero, but was merely converted into the form of potential energy, from which it could be recovered at any time in the kinetic form. We have here an illustration of the extremely important principle of the **Conservation of Energy,** which may be stated as follows:

Energy can neither be created nor destroyed, though it may be converted from one form into another.

Potential and kinetic energy are but two examples of energy in different forms. Other forms of energy are Heat, Light, Sound, Electrical, Magnetic and Chemical Energy. We shall discuss the application of the principle of the Conservation of Energy to these various forms as the occasion arises.

7. Power.

Power is defined as the rate of doing work; the average power over a given time is therefore the total energy expended divided by the time.

8. Motion in a Circle.

In the chapter on *Kinematics* we confined our attention mainly to motion in a straight line. The reason for this was that by Newton's First Law of Motion any departure from rectilinear motion must be ascribed to a force and, therefore, all non-rectilinear motion involves **Dynamics** as well as *Kinematics.* Suppose a body is moving in a circle with uniform velocity v, as shown in fig. 4. Then according to Newton's First Law of Motion there must be a force acting continually on the

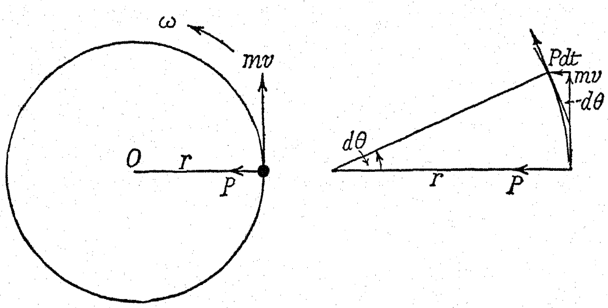

Fig. 4

body to prevent it from flying off at a tangent to the circular path. The simple experiment of whirling a stone on a piece of string held in the hand of the experimenter informs him that the force involved is a tension along the string. The force therefore acts radially. The outward radial force exerted by the body on the hand, through the string, is termed the **centrifugal force.** The force acting on the body is equal and opposite to this, and this inward force on the body is termed the **centripetal force.** To determine the value of this force, suppose the direction of motion is deflected through the small angle $d\theta$ in the short interval of time dt. This deflection is the result of the radial force P, and the momentum imparted radially to the body is, by Newton's Second Law, $P\,dt$. But the tangential momentum of the body is mv. Hence the angle $d\theta$ (fig. 4) through which the direction of motion is deflected in time dt is given by

$$d\theta = \frac{P\,dt}{mv},$$

or

$$P = mv\frac{d\theta}{dt} = mv\omega, \qquad \text{(3.13)}$$

since $d\theta/dt = \omega$, the angular velocity of rotation.

Now, in a circle of radius r, if an arc s subtends an angle θ at the centre, then $s = r\theta$, and therefore $ds/dt = r\,d\theta/dt$, that is, $v = r\omega$. Hence equation (3.13) may be written in either of the important forms

$$\mathbf{P} = m\omega^2 r, \qquad \ldots \ldots \qquad (3.14)$$

$$\mathbf{P} = \frac{mv^2}{r}, \qquad \ldots \ldots \qquad (3.15)$$

where r is the radius of the circle. In our deduction we have neglected the force present due to the weight of the body acting vertically downwards. Our calculation as given above therefore applies only to a body describing a horizontal circle, but the extension to a vertical circle does not involve any radical alteration and is discussed later with reference to the simple pendulum and other examples.

9. Simple Pendulum: Simple Harmonic Motion.

Consider now the motion of a particle, hanging by a light inextensible string. Normally the string hangs vertically at rest. A tension T in the string acts vertically upwards, and is equal and opposite to the weight mg of the particle acting vertically downwards. Suppose the particle is pulled to one side so that the string makes an angle θ_0 with the vertical, and that it is then released. We know by experiment that the particle moves through a circular arc in a vertical plane. Its velocity continually increases as it moves to the bottom of the arc, where it reaches a maximum. It then swings through an angle, which is nearly equal to θ_0, on the opposite side of the vertical, and comes to rest. Next, it returns to its original position, and the motion is continued as before. This apparatus is termed **a simple pendulum.** The motion is a particular case of circular motion, but the considerations of § 8 do not apply directly since, although the path is circular, *the velocity is not constant.* If we time the vibrations with a stop clock we find that **the time required for a complete vibration is always the same.** The circular arc gradually diminishes in length, but the time required for the particle to swing from a position of rest on one side of the vertical, through the vertical to the other side and almost back to its original position, is constant. A vibration having this property is said to be **isochronous.** We shall now proceed to calculate an expression for the time of oscillation of the simple pendulum.

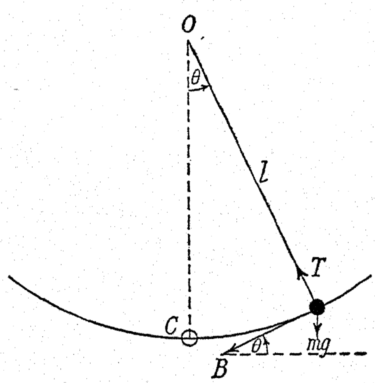

Fig. 5

Referring to fig. 5, we note that the forces on the pendulum are

a tension T up the string, and the weight of the particle, mg, vertically downwards. Since the movement of the particle is along the arc of a circle with centre at O, the tension in the string will produce no acceleration in the direction of motion, for this force acts radially and its component along the arc is zero. The only force causing motion is therefore the component of mg down the direction of the arc. This component is $mg \sin\theta$. Hence the component acceleration down the arc is given by the equation

$$mg \sin\theta = m \times \text{acceleration} = ma$$

or

$$a = g \sin\theta. \qquad (3.16)$$

If the length of the string is l, the tangential acceleration at any instant, since (p. 21) $v = l\omega$ and therefore $dv/dt = l\, d\omega/dt$, is given by the equation

$$a = l \times \text{angular acceleration.}$$

Now

$$l\,d\omega/dt = l\frac{d^2\theta}{dt^2}, \qquad (3.17)$$

since ω, the angular velocity, $= d\theta/dt$.

Note that $+l\ d^2\theta/dt^2$ is the acceleration *in the direction in which θ increases*. Hence, from (3.16) and (3.17),

$$-l\frac{d^2\theta}{dt^2} = g \sin\theta. \qquad (3.18)$$

To find the time of oscillation of the pendulum we clearly have to integrate equation (3.18). Now it is found experimentally that unless θ always remains small the movement of the pendulum is not isochronous and, in fact, is not of special interest. This is fortunate because equation (3.18) is not capable of simple integration. For **small oscillations,** however, we may write $\sin\theta = \theta$, and (3.18) becomes

$$l\frac{d^2\theta}{dt^2} = -g\theta. \qquad (3.19)$$

This equation may be written

$$\frac{d^2\theta}{dt^2} = -p^2\theta, \qquad (3.20)$$

where p^2 is written for g/l and is constant, since g and l are both constant for a pendulum of given length at a particular point on the earth's surface. *Equation* (3.20) *is one of the most important equations in the whole of Physics.* It states that the second derivative of a quan-

tity (θ) is equal to a constant ($-p^2$) times the first power of the quantity. The solution of this equation is

$$\theta = A \sin pt + B \cos pt, \quad . \quad . \quad . \quad . \quad (3.21)$$

where A and B are constants to be found in the way described later. We can prove this directly by differentiating (3.21) twice, thus

$$\frac{d\theta}{dt} = pA \cos pt - pB \sin pt, \quad . \quad . \quad . \quad . \quad (3.22)$$

$$\frac{d^2\theta}{dt^2} = -p^2A \sin pt - p^2B \cos pt. \quad . \quad . \quad . \quad (3.23)$$

Substitute from (3.23) and (3.21) in (3.20), and we get

$$-p^2(A \sin pt + B \cos pt) = -p^2(A \sin pt + B \cos pt),$$

which proves that our solution (3.21) is correct.

From (3.21) therefore, at any time t after the pendulum has been released, the angle at which the string is inclined to the vertical is given by

$$\theta = A \sin pt + B \cos pt.$$

To find A and B we must consider known values of θ and t and substitute them in (3.21). Thus, when $t = 0$, the pendulum has its initial angular displacement, θ_0. Hence (3.21) becomes

$$\theta_0 = B$$

for $\sin pt = 0$ and $\cos pt = 1$ when $t = 0$. To find A we consider equation (3.22). When $t = 0$, the pendulum has no velocity and therefore $d\theta/dt = 0$: hence

$$0 = pA - 0,$$

so that $A = 0$.

The final solution is therefore

$$\theta = \theta_0 \cos pt = \theta_0 \cos\sqrt{\frac{g}{l}}t. \quad . \quad . \quad . \quad . \quad (3.24)$$

Now if the time of a complete oscillation is τ, θ is again equal to θ_0 after a time τ. Substituting in (3.24), we get

$$\theta_0 = \theta_0 \cos p\tau,$$

or

$$\cos p\tau = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.25)$$

Since the cosine equals unity, when the angle is $0, 2\pi, 4\pi$, etc., the first complete oscillation will end when

$$p\tau = 2\pi,$$

or

$$\tau = \frac{2\pi}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}26)$$

3

The period of the pendulum, that is, the time taken to make one complete oscillation, is therefore

$$\tau = 2\pi\sqrt{\frac{l}{g}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.27)$$

for $p^2 = g/l$ by definition.

The **restoring force** acting on the body for any deflection θ is $mg \sin\theta$, which for small angles equals $mg\theta$. This force is therefore directly proportional to the angular displacement θ. Motion in a straight line governed by forces of this type is said to be simple harmonic.

Simple Harmonic Motion is defined as motion in a straight line under a force always directed towards a fixed point in the line, and of magnitude varying directly as the distance from the fixed point.

An alternative definition of simple harmonic motion is that it is *the projection of uniform motion in a circle on to a diameter of that circle.* Thus, in fig. 6, consider a body starting from A, at time $t = 0$, and moving uniformly round a circle with constant angular velocity ω. Then if at any point P of its path we drop a perpendicular on to the diameter AB we obtain the point N. Now it is clear that as P moves round the circle, N moves across the diameter and both points reach B simultaneously. Further, N returns to A by the time P has executed one complete revolution. The point N therefore moves to and fro along AB with the same period as P describes the circle. This period for P is

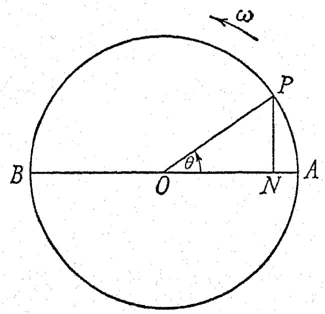

Fig. 6

$$\tau = \frac{2\pi}{\omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.28)$$

If $ON = x$, $OA = a$, and $\angle AOP = \theta$, then $\theta = \omega t$, and

$$x = a \cos\omega t.$$

Hence
$$\frac{dx}{dt} = -a\omega \sin\omega t,$$

and
$$\frac{d^2x}{dt^2} = -a\omega^2 \cos\omega t.$$

$$\therefore \frac{d^2x}{dt^2} = -\omega^2 x.$$

The motion of N is therefore simple harmonic, by the definition, and the acceleration for unit displacement is ω^2. Since $\tau = 2\pi/\omega$, the time of oscillation of a body undergoing simple harmonic motion is the same as that of a body executing a *uniform* circular motion with an angular velocity equal to the square root of the acceleration for unit displacement, that is

$$\tau = 2\pi\sqrt{\frac{1}{\text{acceleration for unit displacement}}}. \qquad (3.29)$$

Other examples of simple harmonic motion are the vertical oscillations of a body suspended on the end of a spiral spring, and the vertical oscillations of mercury contained in a U-tube. We shall discuss the former in detail later (see § 11 (*c*), p. 89); in each case it is usually the period of oscillation which is required, and this may always be determined by finding the acceleration for unit displacement and then substituting in (3.29).

10. Conservation of Momentum.

Consider the mechanical processes involved when a shell is fired from a gun. We know by experience that the gun itself recoils. The question arises as to whether the velocity of the recoil is in any definite way connected with that of the shell. Let the force acting on the shell be P at time t, and let the shell be projected with velocity v, at time T. Then if u is its velocity at any time t less than T, we have

$$P = m\frac{du}{dt},$$

so that

$$\int_0^T P\,dt = \int_0^v m\,du = mv, \quad . \quad . \quad . \quad . \quad (3.30)$$

where m is the mass of the shell. Now by Newton's Third Law of Motion there must at any moment be a reaction on the gun, which at time t is also equal to P. If the gun is situated on ice so that there is negligible friction between it and the ground, and if the shell is fired horizontally, the gun will recoil with a certain velocity V; if its mass is M, exactly the same argument as above gives

$$\int_0^T P\,dt = MV. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.31)$$

Now the L.H.S. of (3.30) and (3.31) are identical. Hence

$$mv = MV, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.32)$$

or the momentum of the shell is equal to the momentum of the recoiling gun. Equation (3.32) may be written

$$mv - MV = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.33)$$

Now before the shell was fired the system was at rest and the momentum was zero. Equation (3.33) shows also that after the shell was fired the *total momentum* of the entire system is again zero. This example illustrates the principle of the **Conservation of Momentum.** In order to demonstrate this important result in the laboratory, a device known as the ballistic pendulum, shown in fig. 7, may be used. Two trays A and B are supported by strings so that the trays are in contact when they are practically horizontal. A is then pulled aside through a given amount, the displacement being observed on a circular scale. When A is released it falls and strikes B and since the edge of A has two small projecting pins and B is edged with cork,

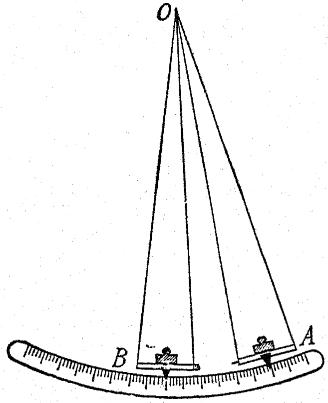

Fig. 7*a*

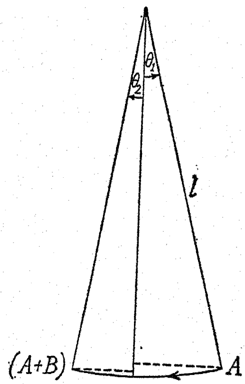

Fig. 7*b*

A and B adhere, and both swing an observed distance along the circular scale. Referring to fig. 7(*b*), if the length of the strings is l and the angles of deflection of the trays before and after collision are θ_1 and θ_2 respectively, then we note that the effective vertical height through which A falls is $l(1 - \cos\theta_1)$, whereas the height to which A and B both rise after collision is $l(1 - \cos\theta_2)$. Applying the principle of the conservation of energy to determine the velocity of A just before impact, we note that the potential energy lost by A is

$$m_A gl(1 - \cos\theta_1) = 2m_A gl \sin^2(\theta_1/2). \quad . \quad . \quad (3.34)$$

This must equal the kinetic energy possessed by A at the lowest point that is, at the instant of impact. Denoting the velocity of A by v_A, and treating the moving masses as material particles, we therefore have

$$\tfrac{1}{2}m_A v_A{}^2 = 2m_A gl \sin^2(\theta_1/2), \quad . \quad . \quad . \quad . \quad (3.35)$$

or

$$v_A{}^2 = 4gl \sin^2(\theta_1/2). \quad . \quad . \quad . \quad . \quad . \quad (3.36)$$

Similarly, if v_{A+B} is the velocity with which A and B commence to move up after the impact

$$v_{A+B}^2 = 4gl\sin^2(\theta_2/2). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.37)$$

Now according to the principle of the conservation of momentum, the momentum before collision is equal to the momentum after collision, that is

$$m_A v_A = (m_A + m_B)v_{A+B}. \quad . \quad . \quad . \quad . \quad . \qquad (3.38)$$

Hence, eliminating v_A and v_{A+B} from (3.37) and (3.38) we obtain

$$\frac{m_A}{m_A + m_B} = \frac{\sin(\theta_2/2)}{\sin(\theta_1/2)}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.39)$$

The experiment is carried out with different weights in the two trays; equation (3.39) is confirmed, and hence the principle of conservation of momentum upon which the experiment rests has been verified.

It should be mentioned that we have considered only the linear momentum of the bodies. If a body is rotating it possesses angular velocity and therefore *angular momentum.* During dynamical processes, under certain conditions, angular momentum is also conserved. We shall discuss this in further detail in the next chapter.

11. Impact of Spheres. Resilience and Restitution.

In the experiment just described with the ballistic pendulum, the bouncing of the tray A from the tray B during impact was avoided by the use of cork and pins. If the same experiment were repeated with the two trays replaced, say, by two steel spheres of different sizes, a certain amount of rebound would occur on impact. By observing the heights to which each sphere rose after impact, and so evaluating the final velocity and hence final momentum of each sphere, the principle of the conservation of momentum could again be tested. It would be found that the *total* momentum after collision was always equal to that before collision, but the distribution of momentum between the two spheres was not that to be expected from their masses and initial velocities. This is explained by assuming that the *resilience* or elasticity of the material enters into consideration. If, for example, a steel sphere approaches normally to the surface of a flat steel slab with velocity u, the velocity of recoil, v, is found to be different from u. The **coefficient of resilience or restitution** for the sphere and plate is usually denoted by e and is defined by

$$e = \frac{\text{velocity of recoil}}{\text{velocity of approach}} = \frac{v}{u}. \quad . \quad . \quad . \quad . \qquad (3.40)$$

An apparatus for measuring the coefficient of restitution is shown in fig. 8. A flat steel plate is mounted horizontally below a short vertical electromagnet M to which may be attached steel spheres S. On breaking the battery circuit to the electromagnet the sphere S drops and rebounds from the plate. The heights of fall and of rebound are determined with a vertical scale and a telescope as shown in the

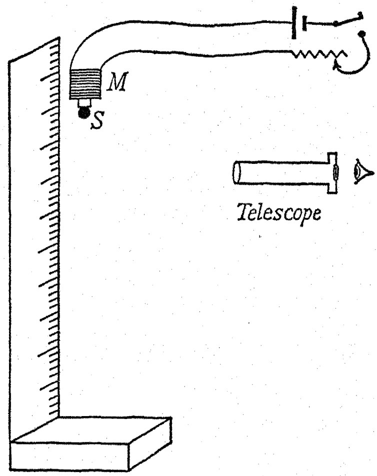

Fig. 8

diagram. Different heights may be taken and the coefficient of restitution will be calculated from the equations

$$\begin{aligned} v_1^2 &= 2gh_1, \\ v_2^2 &= 2gh_2, \\ e &= v_2/v_1. \end{aligned} \qquad (3.41)$$

Hence

$$e = \sqrt{\frac{h_2}{h_1}}. \qquad (3.42)$$

If $e = 1$, the materials are said to be perfectly elastic; for steel e is about 0·95.

EXERCISES

1. State Newton's Laws of Motion and discuss the evidence on which they are based.

2. Define **work**, **energy** and **power**. In what sense can a strong man supporting a heavy weight in a fixed position be said to be doing no work?

3. What is meant by the Principle of the Conservation of Energy? Upon what evidence does it rest?

4. A spring balance hangs from the roof of a stationary lift and supports a weight of 2 Kgm. The lift starts to ascend, whereupon the balance reads 2·5 Kgm. Find the acceleration of the lift. [$\frac{1}{4}g$.]

5. A pile-driver of mass 100 lb. falls freely a distance of 10 ft. vertically on to a pile of mass 1000 lb. and drives it 1 ft. into the ground. Assuming that the pile and driver move together after the impact, find the average value of the resisting force opposing the motion of the pile into the ground. [$1190\frac{10}{11}$ lb.]

6. Determine the least velocity which an aeroplane must have at the top of a circle of radius 1000 ft. if it is to loop the loop successfully. [179 ft./sec.]

7. Show that the energy lost when two smooth spheres collide depends on the square of the coefficient of restitution.

8. A bullet of mass 50 gm. is fired horizontally into a block in which it remains embedded. The block is suspended by a fine string of unknown length to form a simple pendulum. The impact of the bullet causes the block to swing through 2°. If the time of oscillation of the block-pendulum is 10 sec. and its mass is 5 Kgm., find the initial velocity of the bullet. [5500 cm./sec.]

CHAPTER IV

Statics and Dynamics of a Rigid Body

Since a particle at rest has no component acceleration in any direction, it follows from the equation $P = ma$ (p. 15) that **a particle is in equilibrium when the algebraic sum of the component forces**

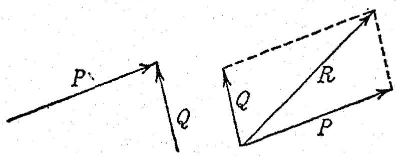

Fig. 1

acting upon it in any direction is zero. Let two forces P and Q meet in a point as in fig. 1. Since forces are vector quantities (p. 8), following the law of vector addition, like velocities and accelerations, the resultant of P and Q is obtained by representing them as the two

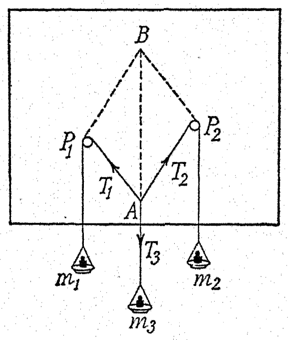

Fig. 2*a*

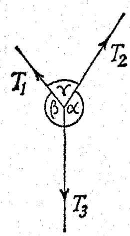

Fig. 2*b*

sides of a parallelogram; the diagonal R, through the point where P and Q meet, is then the resultant. That the **Parallelogram of Forces** is obeyed can be shown directly by experiment using the apparatus shown in fig. 2(a). Two pulleys P_1 and P_2 are mounted on horizontal axes and attached to a vertical board. Masses m_1 and m_2 are attached

to the ends of strings, which pass over pulleys and are attached to a third mass m_3. It is found that equilibrium is attained only when the lengths P_1A, P_2A and AB are proportional to the magnitudes of m_1, m_2 and m_3 respectively; also AB must be vertical. It is clear that half the parallelogram is sufficient to represent the forces completely. The theorem is then known as the **Triangle of Forces.** Since the three sides of a triangle are proportional to the sines of the angles opposite to the sides, consideration of figs. 2(*a*), 2(*b*) shows that

$$\frac{T_1}{\sin\alpha} = \frac{T_2}{\sin\beta} = \frac{T_3}{\sin\gamma}, \quad . \quad . \quad . \quad . \quad . \quad (4.1)$$

so that each force is proportional to the sine of the angle between the other two.

1. Equilibrium of a Rigid Body.

Suppose we have a rigid body of finite extent such as the uniform bar shown in fig. 3. If this bar is supported by a horizontal axis perpendicular to the plane of the paper, and passing through its centre

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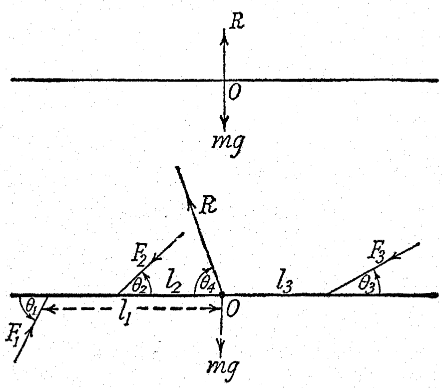


Fig. 3

at O, we should find experimentally that the bar would be in equilibrium in any position. There must clearly be a vertical reaction R exerted by the axis on the bar, to balance its weight mg acting vertically downwards. Suppose now forces F_1, F_2, F_3, *all in a vertical plane*, act as shown at points l_1, l_2 and l_3 from O in directions inclined at θ_1, θ_2 and θ_3 to the length of the bar. What are now the conditions for equilibrium? We shall first state the conditions and then explain and justify our statements:

(*a*) **The algebraic sum of the forces in any direction must be zero.**

(*b*) **The algebraic sum of the moments of the forces about any point in the plane containing the forces must be zero.**

(For definition of *moment of a force* see below.)

The first condition must be satisfied if there is to be no *translational* motion. The reaction R will no longer act vertically as it did before, but will be inclined at some angle θ_4 to the bar. To determine θ_4, we resolve the forces horizontally and vertically and equate the sum of each set to zero. It is usually more convenient to resolve horizontally and vertically, although any two directions mutually at right-angles will suffice. Two directions are necessary to ensure that there is no resultant force at right angles to the direction first considered. Thus we have

$$F_1 \sin\theta_1 + R\sin\theta_4 = F_2 \sin\theta_2 + F_3 \sin\theta_3 + mg, \quad . \quad (4.2)$$

and

$$F_1 \cos\theta_1 - R\cos\theta_4 = F_2 \cos\theta_2 + F_3 \cos\theta_3. \quad . \quad . \quad (4.3)$$

These equations are sufficient to determine the values of R and of θ_4.

The second condition is required to ensure that there shall be no *rotational* movement of the body.

The moment of a force about a point is defined as the product of the force and the perpendicular distance from the point to the line of the force.

The moment of a force about *an axis, to which the direction of the force is perpendicular,* is defined as the product of the force and the length of the common perpendicular to the line of the force and to the axis.

A system consisting of two equal and parallel forces in opposite directions is called a **couple.** We see at once that the sum of the moments (taken with opposite signs) of the two forces is the same about all points in their plane, being equal to the product of either force by the perpendicular distance between them. This constant algebraic sum of the moments is called the **moment of the couple,** or simply the **couple.**

Taking moments of the forces about O, or about the axis through O, in fig. 3, we have

$$F_1 l_1 \sin\theta_1 - F_2 l_2 \sin\theta_2 + F_3 l_3 \sin\theta_3 = 0, \quad . \quad . \quad (4.4)$$

for no rotational movement. Note that by taking moments about O, the contribution of R is eliminated, since for this force the perpendicular distance is zero. The same considerations apply to a body of irregular shape; we shall consider this more general case later, prior to our discussion of the compound pendulum (see Chap. VI). However, we may note here that the concept of the centre of gravity of a body arises when we consider the resultant of the forces on the body due to the weights of the innumerable small parts of which the body

may be considered to consist. Thus, the irregular body shown in fig. 4 experiences an infinite number of parallel forces each equal to $\Delta m \,.\, g$, due to the action of gravitational attraction. The sum total of these forces is the weight of the body, and this will act at some point termed the **Centre of Gravity.** The position of the centre of gravity is determined by the condition that a body suspended by an axis passing through this point exhibits no tendency to rotate, for since the perpendicular distance of the resultant from this point is zero, the moment about this point is zero. On the other hand, if the body is freely suspended about any other point, as shown in fig. 5, since the total weight

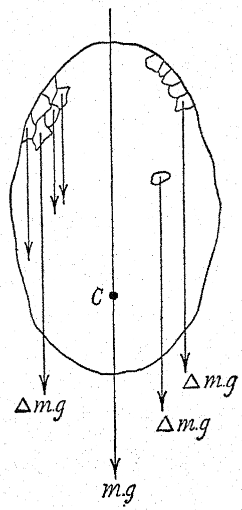


Fig. 4

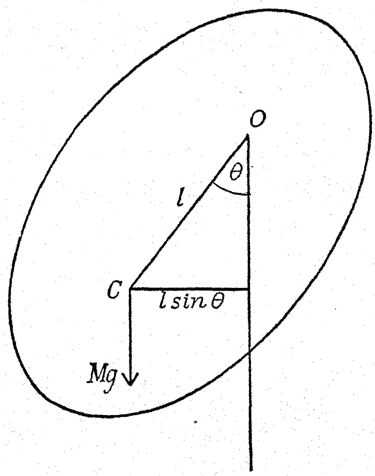


Fig. 5

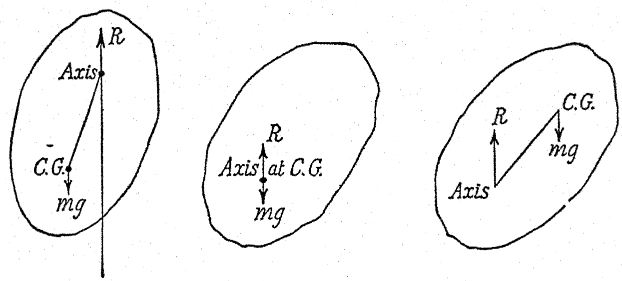
of the body may be considered as concentrated at the centre of gravity, there will be a turning moment about the point of suspension given by $Mgl \sin\theta$. The body will therefore turn until this turning moment is zero, that is, θ is zero. This will occur when l is vertical. Hence **the centre of gravity of a suspended body always lies in the vertical line containing the point of suspension.** This fact leads to a simple method of determining the centre of gravity of such a body as an irregularly shaped lamina; for if the latter is suspended from two points successively, the centre of gravity must be situated at the point of intersection of the two vertical lines which contained the points of suspension. We may further note that the body is in equilibrium for three positions, namely:

(*a*) with the centre of gravity vertically *below* the point of suspension. This position is said to be a position of **stable equilibrium,**

since if the body is slightly displaced, the turning moment about the axis of suspension is such as to cause the body to return toward its initial (equilibrium) position;

(*b*) with the centre of gravity coinciding with the point of suspension. This is said to be a position of **neutral equilibrium,** since if the body is rotated there is no tendency for it either to return or to rotate further. It is in equilibrium for all orientations;

(*c*) with the centre of gravity vertically above the axis of suspension. In this position the equilibrium is said to be **unstable,** since



Stable Neutral Unstable

Fig. 6

if the body is given a slight displacement, the moment of the weight about the axis of suspension will cause the body to move farther and farther away from its initial position of unstable equilibrium until it finds its position of stable equilibrium, when its centre of gravity lies directly below the axis of suspension. These ideas are illustrated in fig. 6.

2. Machines.

We are now in a position to consider the action of a great many simple machines which are based upon the principles which we have considered.

(i) The Roman Steel-yard and the Danish Steel-yard.

These consist of uniform horizontal rods or **levers** which are used for weighing. The former has a movable counterpoise and fixed fulcrum and is calibrated by placing known masses M in the scale pan (see fig. 7) and altering the position of the counterpoise C until the moments of the two weights about the fulcrum are equal and the rod is horizontal. The value of the weight in the scale-pan is engraved on the lever at the appropriate position of the counterpoise, when balance is obtained.

With the *Danish steel-yard*, the counterpoise is fixed, consisting of

a heavy sphere S, as shown in fig. 8; the fulcrum F is movable, and thus allows a balance to be obtained for various weights in the scale-

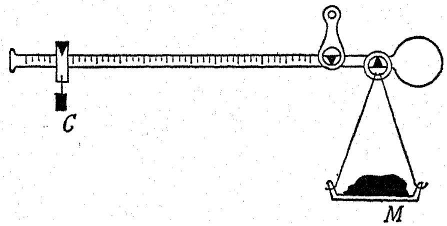


Fig. 7

pan. The appropriate weight for balance is engraved on the lever for various positions of the fulcrum.

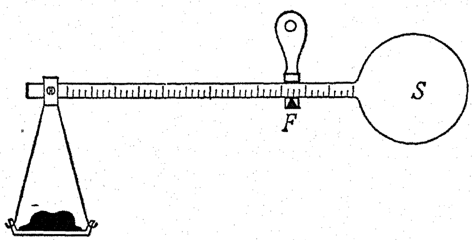


Fig. 8

(ii) **The Common or Chemical Balance.**

The principle of the common or chemical balance is shown in fig. 9, which shows a diagram of a modern instrument. It consists in principle of a horizontal lever balanced on a knife-edge at its centre. The lever is of compound construction, so that the centre of gravity of the system lies below the horizontal axis of suspension. In fig. 10 the balance point is at O and the centre of gravity at G. Normally, the balance arm is horizontal, in the position AB, but if unequal weights are placed in the scale-pans the arm takes up some position PQ lying at an angle θ to AB. Suppose now weights W_1 and W_2 are placed in the scale pans and the balance arm is turned through an angle θ, in which position it is in equilibrium. Then if the weight of the balance arm is W, the condition for equilibrium, as found by taking moments about O, is

$$W_1 l \cos\theta = Wh\sin\theta + W_2 l \cos\theta; \quad . \quad . \quad . \quad (4.5)$$

hence

$$\tan\theta = \frac{(\mathbf{W}_1 - \mathbf{W}_2)}{\mathbf{W}}\frac{l}{h}, \quad . \quad . \quad . \quad . \quad (4.6)$$

where l is half the length of the balance arm and h is the distance OG.

The properties of a good balance are as follows:

(a) **The balance arm should be horizontal when equal weights are placed in the two scale-pans.** This implies that the fulcrum must be exactly at the centre of the balance arm and that the mass of the

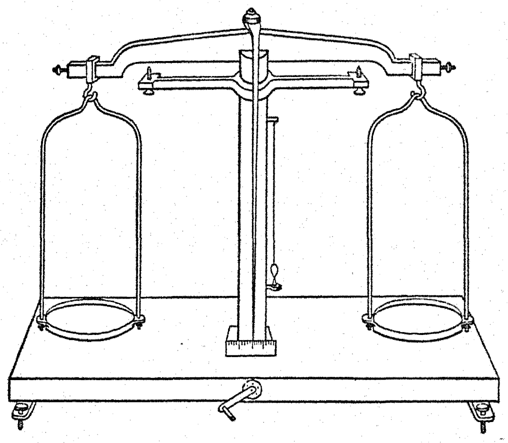


Fig. 9

latter must be distributed quite symmetrically on either side of the fulcrum. If this condition is not fulfilled, the correct mass of a body is given by $(W_1W_2)^{\frac{1}{2}}$, where W_1 and W_2 are its apparent weights when it is weighed first in one scale-pan and then in the other. The proof of this is left to the reader as an exercise.

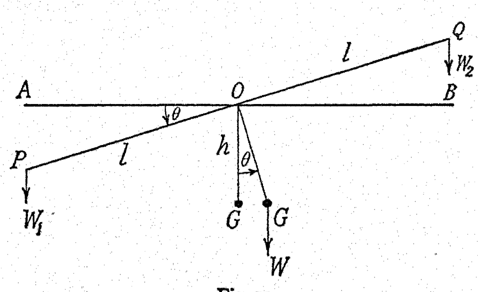


Fig. 10

(b) **The balance should have high sensitivity;** that is, it should register a large deflection for a small difference in the weights in the two scale-pans. The condition for high sensitivity is found from equation (4.6), which shows that θ is a maximum for a given value of $(W_1 - W_2)$ when l is a maximum and W and h are as small as possible. Hence for a high sensitivity the balance arm should be as long as

possible but as light as possible, while the centre of gravity should be only just below the knife-edge.

(c) **The balance should be robust**; that is, it should not bend under moderate loads. This condition requires a short thick balance arm, and therefore is diametrically opposed to the conditions for maximum sensitivity. In practice a compromise must be effected, but the history of the balance shows a series of improvements with the invention of light but rigid metal alloys.

(d) **The balance should be quick in action.** This implies that the period of the oscillation which occurs when the balance is released should be as short as possible. The requirements of a short period of swing are precisely the reverse of those for high sensitivity, so again a compromise must be effected.

(e) **It should return to zero when the weights are removed**; this implies that the balance must be a system in stable equilibrium.

(f) **The sensitivity should be as far as possible independent of the load in the scale-pans.** This can only be achieved if the knife-edges which support the scale-pans and the knife-edge which constitutes the fulcrum are parallel and all lie in one horizontal plane. Otherwise, as the tilt of the beam is increased, the effective length of the arm supporting the lower pan is decreased; an additional restoring couple is therefore called into play, given by $W(l_1 - l_2)$, where W is the value of each weight in the scale-pan and l_1 and l_2 are the projected horizontal lengths of the two halves of the beam. This couple increases as W increases, and hence the sensitivity will decrease as the load in the scale-pan is increased.

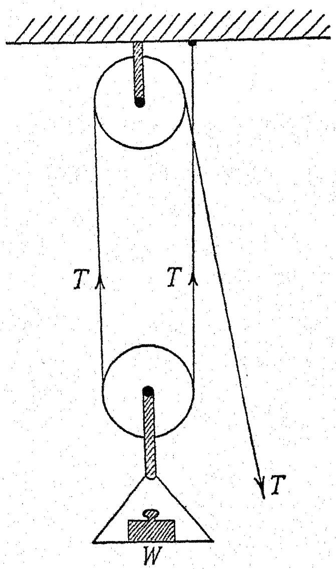


Fig. 11

(iii) Pulleys.

If we consider the simple arrangement of pulleys shown in fig. 11, we observe that the **load** or weight W is supported by the two tensions T. The latter are transmitted from the tension or **effort** applied by the operator and hence the load W is supported by an effort only half its magnitude. This ability to move weights greater than the effort applied is called the **mechanical advantage** of the machine, and is defined as the ratio of the load supported to the effort applied. We observe, however, that (not only in fig. 11, but generally) if the effort is moved through

a distance a, the load will move through a distance b given by

$$Wb = Ta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.7)$$

This follows from the conservation of energy, the potential energy gained by the load equalling the work done by the effort. Since (4.7) may be written

$$\frac{a}{b} = \frac{\mathrm{W}}{\mathrm{T}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.8)$$

we see that the distance moved by the effort is greater than that moved by the load in the ratio of load to effort. The ratio a/b in (4.8)

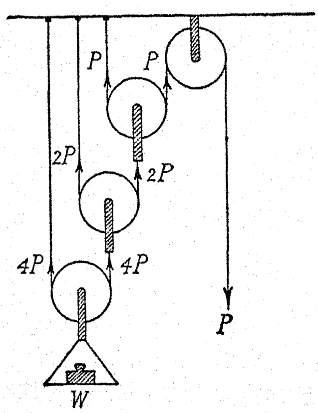


Fig. 12*a*

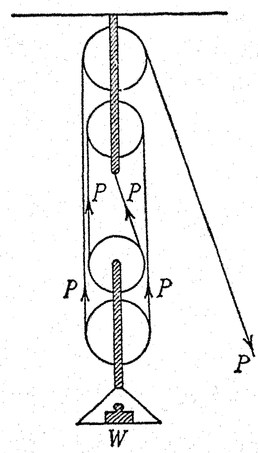


Fig. 12*b*

is known as the **velocity ratio**; it is obviously identical with the mechanical advantage, under the conditions we have assumed. Should friction be present the equations must be modified, as the effort will have to do additional work to overcome the friction as well as raise the load.

Other systems of pulleys are shown in figs. 12(a) and 12(b), the former being known as the **Archimedean, or First System of Pulleys,** and the latter as the **Second** or **Common System of Pulleys.** To avoid the long length of rope required in the system last discussed, a modification termed **Weston's Differential Pulley** is used. As shown in fig. 13, this consists of a lower movable block containing an ordinary pulley and an upper fixed block containing two toothed pulleys of radii r and R respectively. A single endless chain passes round the pulleys. If the effort applied is P, then by considering the upper block

$$PR = TR - Tr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.9)$$

while for the lower block

$$W = 2T. \quad \ldots\ldots\ldots \quad (4.10)$$

Hence the **mechanical advantage is**

$$\mathbf{\frac{W}{P} = \frac{2R}{R - r}}. \quad \ldots\ldots \quad (4.11)$$

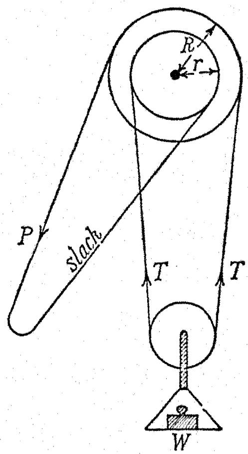


Fig. 13

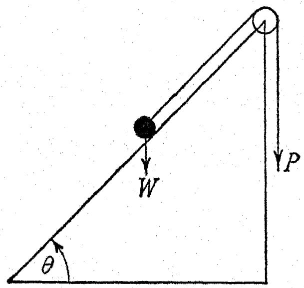


Fig. 14

(iv) The Inclined Plane.

Referring to Chap. III, § 5, we note that a weight P acting parallel to a smooth plane, and given by

$$P = W \sin\theta, \quad \ldots\ldots \quad (4.12)$$

is sufficient to support a weight W resting on the plane. The mechanical advantage of such a system is therefore cosec θ. In practice the effort is applied vertically downwards and transmitted over a pulley at the top of the plane as shown in fig. 14.

(v) The Screw.

A screw may be regarded as an inclined plane which has been wrapped continuously around a vertical cylinder. In practice it is usually encountered in the form of a jack. In this machine an effort P is applied horizontally to the screw at a distance r from the axis of the screw, which is vertical (see fig. 15). The screw works upwards in a fixed nut, and the load W, which rests on the screw, is thereby raised. From the conservation of energy, the work done by the effort equals the gain in potential energy of the load, so that if the pitch of the screw is h,

$$2\pi r P = Wh.$$

Hence the **mechanical advantage** is

$$\frac{\mathrm{W}}{\mathrm{P}}=\frac{2\pi r}{h}; \qquad \ldots\ldots \quad (4.11a)$$

that is, it is equal to the ratio of the circumference of the cylinder to the pitch of the screw. In practice, the friction between the nut and

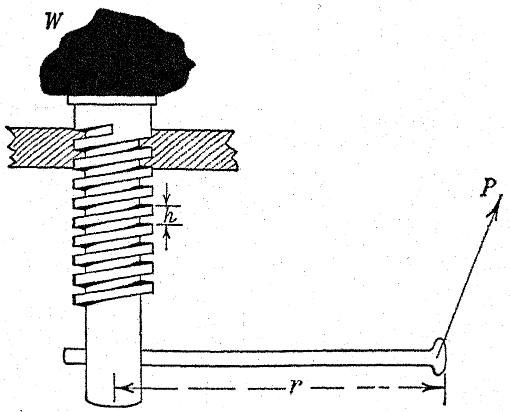


Fig. 15

the screw is often large and the mechanical advantage is then only a small fraction of that given by (4.11a).

3. Friction.

If a small body is placed on a plane inclined at a small angle θ to the horizontal, then although a force $mg\sin\theta$ acts down the plane, m being the mass of the body, the latter does not in general move down the plane. There must therefore be called into play an equal and opposite force opposing the motion. This is termed the *frictional force*; and friction may be defined as the force tending to prevent one surface moving over another. The subject was examined experimentally by Coulomb with an apparatus of the type shown in fig. 16.

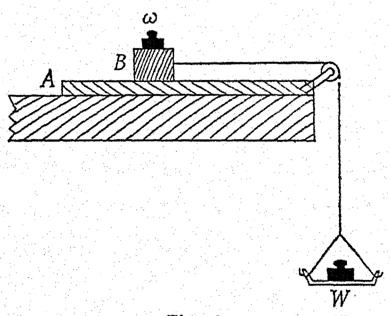


Fig. 16

The two bodies between which the friction is to be examined are made into the shape of two flat slabs A and B. One of the slabs A is clamped to a horizontal table, and the other slab B is attached by a

string to a scale-pan into which various weights may be placed. The slab B is laid on A, and a horizontal tension is applied to B by the string, which passes over a pulley at the edge of the table. It is then found that if B is loaded with various weights w, certain other weights W must be placed in the scale-pan in order that sliding may just commence. In this way Coulomb was able to enunciate the **laws of static friction,** which are:

(1) **The frictional force just before motion commences, or the limiting friction, is directly proportional to the normal reaction between the two surfaces.**

(2) **The limiting friction is independent of the area of contact, provided the normal reaction remains unchanged.**

(3) **The coefficient of limiting (or static) friction,** denoted by μ, is given by

$$\mu = \frac{\mathrm{P}}{\mathrm{R}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.13)$$

where P is the limiting friction and R is the normal reaction.

It follows that if a small body is resting on an inclined plane, the angle of which is gradually increased, slipping will occur for some value known as the **angle of friction** or **angle of repose,** α, given by the relation

$$P = \mu R = mg \sin \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.14)$$

But $$R = mg \cos \alpha,$$

and therefore $$\mu = \tan \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.15)$$

It should be realized that the full frictional force or limiting friction is not called into play until the condition is reached when the body is about to move. When movement has started, the frictional force, which is then termed the **kinetic friction,** is found to be considerably less than the limiting static friction. For velocities which are not too large, the kinetic friction is found to be independent of the velocity and proportional to the normal reaction. We therefore have the relation

$$\nu = \frac{\mathrm{P}}{\mathrm{R}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.16)$$

defining ν, the coefficient of kinetic friction.

To examine kinetic friction and to determine the value of the coefficient ν, Perry's apparatus (fig. 17) may be used. This consists essentially of a vertical axle AB, to which is attached rigidly a wheel C, on the upper surface of which a circular slab of one of the materials F may be clamped. The other surface G rests on F, and is held in contact with it by a normal reaction supplied through the lever AS, the

fulcrum of which is at A, by the variable effort, represented by the loaded scale-pan, at S. As the wheel is rotated, movement of G is prevented by a tangential force supplied by a string, which lies hori-

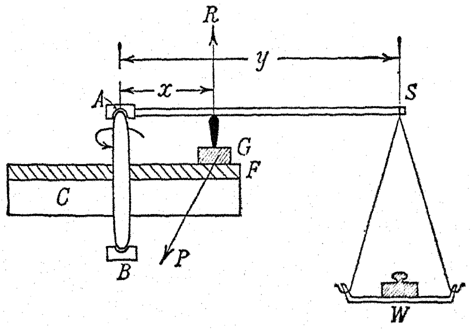


Fig. 17

zontally and passes over a pulley, and is ultimately attached to a scale-pan containing varying weights. The tension in the string then gives the value of P for equation (4.16) while the value of R is given

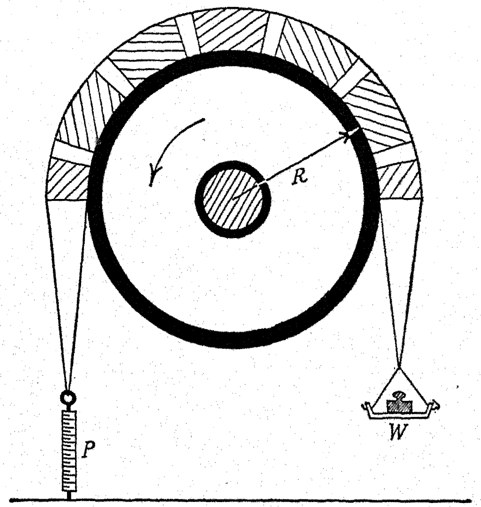


Fig. 18

by Wy/x, where x and y are the lengths indicated in fig. 17. The condition aimed at is that of floating equilibrium, in which the surface G stays approximately in one position as the surface F is revolved at at uniform speed.

In the vast majority of cases friction represents lost energy, and is therefore kept as low as possible. One important example of how friction has been turned to account is the **friction dynamometer.** This instrument is used to measure the horse-power of machines, the measure of which as obtained in this manner is sometimes referred to as the brake-output. A friction dynamometer, as shown in fig. 18, consists of a large wheel, which may be a pulley wheel or the flywheel of the machine concerned, over which passes a flexible belt attached to which are wooden brake blocks. One end of the belt is attached to a spring balance which is fixed rigidly to a beam; the other end supports a weight W. If the spring balance registers an amount P when the wheel is rotating uniformly at a rate of n revolutions per second, then **the rate at which work is being done is given by $2\pi n\mathbf{R}(\mathbf{W} - \mathbf{P})$,** where R is the radius of the wheel. The force $(W - P)$ is, of course, just balancing the frictional force which is being called into play in the condition of floating equilibrium.

4. Rotation of Rigid Bodies.

In Chap. III, § 2, we defined the mass of a body as a quantity associated with a body which is inversely proportional to linear acceleration when a given force is applied to it. Suppose now we apply a couple G (p. 32) to a rigid body which is free to rotate about a fixed axis. We shall show that there is a quantity, associated with the body and this axis, which is inversely proportional to the angular acceleration, about the axis, produced by the couple. The quantity is called the **moment of inertia** of the body about the axis. Expressing this mathematically, we have, denoting the moment of inertia by I and the angular acceleration by $d^2\theta/dt^2$,

$$G = I\frac{d^2\theta}{dt^2}. \qquad \ldots\ldots \qquad (4.17)$$

We shall now examine this formula, and at the same time find an expression for the value of I.

If we concentrate upon what is occurring to each of the small elements which make up the rigid body, we realize that each is experiencing a force urging it into rotational motion about the axis. If for any elementary particle of mass dm, situated a distance r from the axis, the force is P, and G is the *external* couple (couples due to *internal* forces cancelling out, by Newton's Third Law), then

$$\begin{aligned} G &= \Sigma Pr = \Sigma(dm\,.\,\text{linear acceleration})r \\ &= \Sigma\left(dm\,.\,r\frac{d^2\theta}{dt^2}\right)r \\ &= \Sigma(dm\,.\,r^2)\frac{d^2\theta}{dt^2}. \qquad \ldots\ldots\ldots \qquad (4.18) \end{aligned}$$

Comparing (4.17) and (4.18) we see that the moment of inertia I is given by the expression $\Sigma(dm \cdot r^2)$. This last expression may be written in integral form and integrated for certain special cases, but for bodies of irregular shape resort must be made to experiment, in which all the quantities may be measured except I, and hence I determined. An additional quantity of considerable use is the **radius of gyration** of a rigid body about an axis. This quantity is defined as the linear quantity k, where

$$k^2 = \frac{\Sigma(dm \cdot r^2)}{\Sigma dm}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.19)$$

It follows from (4.19) that the moment of inertia

$$I = \Sigma(dm \cdot r^2) = \Sigma dm \cdot k^2 = Mk^2$$

where M is the mass of the body. In certain circumstances it is preferable to use the form Mk^2 instead of I.

If we write (4.17) in the form

$$G = I \times \text{angular acceleration} = I \times \frac{d\omega}{dt} = \frac{d}{dt}(I\omega), \qquad (4.20)$$

where ω is the angular velocity of rotation, we note that there is a formal resemblance with equation (3.5),

$$P = \frac{d}{dt}(mv).$$

By analogy with the quantity mv which is termed the linear momentum of a body, the quantity $I\omega$ is termed the **angular momentum.** The analogy is more than formal, for experiment shows that, just as for linear momentum, angular momentum is conserved in a rotating system, not subjected to external force. This also follows from (4.20), which shows that $I\omega$ is constant if $G = 0$.

Finally, in complete analogy with the expression for kinetic energy of translation $\frac{1}{2}mv^2$, we have the quantity $\frac{1}{2}I\omega^2$ which represents the **kinetic energy of rotation.** To show this more fully, let a particle of mass m be rotating about an axis at a distance r from the particle, and let the angular velocity of rotation be ω. Then the linear velocity of the particle is $v = r\omega$, so that the kinetic energy of the particle is $E = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$. If we consider a solid body as made up of a multitude of such particles, the total kinetic energy of the rotating body is $\Sigma\frac{1}{2}mr^2\omega^2 = \frac{1}{2}\omega^2\Sigma mr^2$. Now the summation term is, by definition, the moment of inertia I of the body about the axis of rotation, so the kinetic energy of rotation is $\frac{1}{2}I\omega^2$. This form of energy also obeys the general law of conservation of energy.

5. Oscillation of Rigid Bodies.

Consider the irregular rigid body shown in fig. 5 (p. 33). A horizontal axis supports the body and in the equilibrium position the centre of gravity of the body will lie directly below the axis of suspension. Let the body now be rotated slightly to one side so that the line joining the axis of suspension to the centre of gravity makes an angle θ with the vertical. Then if the body is released, the centre of gravity will describe a circular arc of radius l with the axis of suspension as centre. The problem is clearly analogous to that of the simple pendulum; and this arrangement using a rigid body is termed a **compound** or **rigid pendulum.** The couple acting about the axis of rotation is $Mg \,.\, l \sin\theta$ and hence, applying (4.17), the angular acceleration with which the body moves is given by

$$G = Mgl \sin\theta = -I\frac{d^2\theta}{dt^2}, \quad . \quad . \quad . \quad . \quad (4.21)$$

where I is the moment of inertia about the axis of rotation. (For the negative sign, compare pp. 22, 24.)

For small angles, (4.21) may be written

$$I\frac{d^2\theta}{dt^2} = -Mgl\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.22)$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{Mgl}{I}\theta.$$

Now equation (4.22) is precisely similar to equation (3.20) where $p^2 = Mgl/I$. Hence, by (3.26), the time of oscillation of a compound pendulum is

$$t = 2\pi\sqrt{\frac{I}{Mgl}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.23)$$

If we express the moment of inertia in the form Mk^2 (4.23) becomes

$$t = 2\pi\sqrt{\frac{k^2}{gl}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.24)$$

EXERCISES

1. State the conditions for the equilibrium of a rigid body.

A uniform ladder rests with its upper end against a smooth wall and its lower end in contact with a rough horizontal plane. If the coefficient of friction is 0·5, find the angle at which the ladder is inclined to the vertical when it is on the point of slipping. [45°.]

2. Explain carefully what is meant by the Centre of Gravity of a body.

Find the position of the C.G. of a geometrically uniform bar which increases in density at a rate proportional to its distance from one end. [$\frac{2}{3}l$.]

3. Describe the construction of a sensitive chemical balance and prove that the sensitivity is independent of the linear dimensions for geometrically similar balances of the same material.

4. Explain clearly what is meant by the moment of inertia of a rigid body about an axis.

A hoop rests in a vertical plane with its inner surface supported by a small horizontal peg. If the period of a small oscillation in a vertical plane is 3 sec., find the radius of gyration of the hoop about its centre, given that the latter is 3 ft. away from the peg. [3·6 ft.]

5. Give practical examples of the conservation of linear and of angular momentum.

If the earth suddenly contracted to half its present radius, by how much would the day be decreased? [18 hours.]

6. Explain the action of (*a*) Weston's Differential Pulley, (*b*) a car jack.

Find the effort required to lift a weight of one ton with a jack whose pitch is $\frac{1}{2}$ in. if the effort is applied 4 ft. from the axis and the efficiency is 75 per cent. [4·95 lb.]

7. Describe an experiment for measuring the coefficient of kinetic friction.

If a bobsleigh slides at constant speed down a slope inclined at 20° to the horizontal, determine the coefficient of friction which is operative, and assuming this remains constant, determine the acceleration of the bobsleigh down a slope inclined at 45°. [0·364; 0·45g.]

CHAPTER V

Units and Dimensions

1. Introduction.

We have proceeded so far on the assumption that the reader's everyday conception of what is meant by length, mass and time is sufficient to allow the general argument to be followed. We must now be more precise, introducing definite **units** and considering the **dimensions** of physical quantities.

2. Fundamental Units.

The statement that a given rod has a length of 10 feet implies that a given standard or unit of length, the foot, has been chosen and that ten of these units can be placed end to end along the length of the rod. The unit of length chosen, and indeed any unit, should have the following properties: it should be

(1) *well-defined*;
(2) *not subject to secular* (*time*) *change*;
(3) *easily compared with similar units*;
(4) *easily reproduced.*

Two systems of units which are universally recognized are termed the **British system** and the **French** or **metric system.**

The British unit of length is the **yard,** which is defined as the straight distance between the transverse lines in two gold plugs on a bronze bar kept at 62° Fahrenheit and preserved in the Exchequer office. We must emphasize that the choice of such a length was quite arbitrary. It is inconveniently large for some purposes, and is sub-divided into three equal parts, each of which is termed a **foot.** The unit of length in most other countries is the metre. This is the length of a bar, originally made to be as nearly as possible equal to one ten-millionth part of a quadrant of the earth, passing through Paris. This relation was subsequently found to be inaccurate, but the original bar was retained, and the **metre** is now defined as the quite arbitrary distance between two marks on a platinum bar kept at 0° C. and preserved in the International Bureau of Standards at St. Cloud, near Paris. The metre, which is slightly more than a yard, is also inconveniently large and has been divided into 100 equal parts, each of

which is called a **centimetre.** The British unit of mass is also quite arbitrary. It is termed the **pound avoirdupois,** and is the mass of a piece of platinum preserved in the office of the Exchequer and marked "P.S. 1844, 1 lb." It bears no simple relation to the unit of volume (the cubic foot) on the same system, and thus differs from the unit of mass in the metric system. This, the **kilogramme,** was initially made as close as possible to the weight of 1000 cubic centimetres of pure water at its temperature of maximum density. Although subsequent work has shown that the relation is not quite accurate, the original kilogramme has been retained and is now simply taken as the weight of a piece of platinum preserved at the Bureau of Metric Standards.

The British and metric systems agree in taking for the unit of time, the **mean solar second.** This is simply the **mean solar day** divided by 86,400. In practice, the mean solar day is based on the **sidereal day.** The latter is defined as the time taken between two successive passages of the same fixed star across the meridian of the place of observation. The solar day is longer than the sidereal day, for between two successive passages over the meridian the sun moves backwards from west to east relative to the fixed stars. A year contains 366·25 sidereal days, but only 365·25 solar days, and hence the mean solar second is 366·25/365·25 of the sidereal second.

For scientific use the metre and the kilogramme are subdivided into one hundred equal parts, and the **centimetre-gram-second** or **C.G.S.** system is obtained. Unfortunately, in engineering, larger units are sometimes considered preferable and, to add to the confusion, that adopted by British countries is the **foot-pound-second** or **F.P.S.** system. The units of length, mass and time are often referred to as **fundamental units,** for reasons which we shall discuss in the next section. We should emphasize here that there is nothing fundamental about them in the sense that Nature considers them more important than other physical concepts such as velocity, electric current, &c. But the various physical quantities such as velocity, force, length, acceleration, mass, energy, and so on, are connected by definitions and physical laws, and once the units of a certain number of these quantities have been chosen as fundamental units, the units of all other quantities can be expressed in terms of them.

3. Derived Units.

Consider now the expression for the area of a surface. The unit in which the area is expressed is the area of a square whose side is the unit of length. Similarly, the unit of volume is that of a cube whose side is the unit of length. It is clear therefore that when the unit of length has been fixed, those of area and volume become fixed automatically. The latter are therefore examples of **derived units,** being

derived from the arbitrarily chosen *fundamental unit* of length. Consideration of velocity takes us a stage farther. Velocity is rate of change of position, and hence will be expressed as length divided by time. Hence, velocity is expressed in derived units based on feet per sec. or cm. per sec., according as the F.P.S. or C.G.S. system is used to define the fundamental units. Again, since force may be defined by equation (3.2) of Chap. III, namely, $P = ma$, force will be expressed in lb. $\times$ ft./sec.^{2} or gm. $\times$ cm./sec.^{2}, the former being given the name **poundals** and the latter, **dynes.** Since $P = 1$ if $m = a = 1$, the **dyne is that force which will produce an acceleration of one cm. per second^2 when it acts on a mass of one gramme.** Similarly, the **poundal** may be defined as **that force which, acting on a mass of one pound, produces in it an acceleration of one foot per second.^{2}**

Other important units are given in the following Table.

		C. G. S. System		British System	
Quantity	Definition	Units	Name	Units	Name
Kinetic Energy	$\frac{1}{2}mv^2$	$m = 1$ gm. $v = 1$ cm./sec.	Erg	$m = 1$ lb. $v = 1$ ft./sec.	Foot-poundal
	or $\dfrac{\frac{1}{2}mv^2}{g}$	$\begin{cases} m = 1 \text{ gm.} \\ v = 1 \text{ cm./sec.} \\ g = 981 \text{ cm./sec.}^2 \end{cases}$	Gm.-cm.	$m = 1$ lb. $v = 1$ ft./sec. $g = 32$ ft./sec.^{2}	Foot-pound
Potential Energy or Work	mgh	$\begin{cases} m = 1 \text{ gm.} \\ h = 1 \text{ cm.} \\ g = 981 \text{ cm./sec.}^2 \end{cases}$	Erg	$m = 1$ lb. $h = 1$ ft. $g = 32$ ft./sec.^{2}	Foot-poundal
	mh	$m = 1$ gm. $h = 1$ cm.	Gm.-cm.	$m = 1$ lb. $h = 1$ ft.	Ft.-lb.
Power	$\dfrac{mh}{t}$	$\begin{cases} m = 1 \text{ gm.} \\ h = 1 \text{ cm.} \\ t = 1 \text{ sec.} \end{cases}$	Gm.-cm./sec.	$m = 1$ lb. $h = 1$ ft. $t = 1$ sec. One Horse-power $= 550$ ft.-lb./sec.	Ft.-lb./sec.

Up to the present, all the quantities considered have been expressed in terms of either length, mass or time or of combinations of these. Such a simple and satisfactory state of affairs continues to hold while we are concerned solely with the mechanical properties of bodies. When, however, we come to consider other physical phenomena such as heat, light, electricity and magnetism, matters become more complicated. In particular it becomes necessary to introduce other fundamental units in addition to those of mass, length and time. Examples of the procedure in such cases are given as occasion arises.

4. Dimensions.

Consider equation (3.27), Chap. III, which states that the time of oscillation of a simple pendulum is given by

$$t = 2\pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum and g is the acceleration due to gravity. Now this is an **equation,** so that not only must the numerical value of both sides of the equation be the same, but also the physical quantities expressed on the two sides must be identical. On the L.H.S. the only quantity is the time, and to denote that a measurement of this physical phenomenon is required we represent it by T. Now for equation (3.27) to be true, the R.H.S. must somehow also be expressed simply as a time measurement T. As for the individual quantities composing the expression on the R.H.S., 2 and π are simply numbers and need not be considered as far as physical measurement is concerned. On the other hand, the length of the pendulum l involves a length measurement which we shall symbolize by L, while the acceleration due to gravity involves a measurement of length and time, the nature of which is expressed by L/T^2 or LT^{-2}. These symbols L and T, which denote the type of physical measurement which has to be made, are termed the **dimensions** of the quantities involved, and the compound expressions such as LT^{-2} and MLT^{-2} for acceleration and force respectively are referred to as the **dimensional formulæ** of acceleration and force. Again, acceleration and force are said to have dimensions of 1 in L, -2 in T; and 1 in M, 1 in L and -2 in T respectively. Substituting the dimensional formulæ of length and acceleration on the R.H.S. of equation (3.27), we obtain $\sqrt{\frac{L}{LT^{-2}}}$, or T, which shows that the R.H.S. of the equation is expressed by T, as well as the L.H.S. The fact that both sides of a physical equation have the same dimensions may be stated as follows: **Equations representing possible physical phenomena must be dimensionally homogeneous.** This property is extremely useful, as it sometimes enables us to predict the equations governing physical phenomena before we have determined them by experiment. As an illustration, suppose we wished to know how the time of oscillation t of a simple pendulum depended on its length l, the acceleration due to gravity g and the mass m of the pendulum bob. Then we assume that

$$t = f(m, l, g), \quad \ldots \ldots \quad (5.1)$$

where $f(m, l, g)$ is some function of the quantities involved. We then

assume that this function may be expressed as an algebraic formula thus

$$t = m^{\alpha} l^{\beta} g^{\gamma}, \quad \ldots \ldots \quad (5.2)$$

where α, β and γ are the powers to which the quantities m, l and g are raised in the final expression. Now if equation (5.2) represents a possible state of affairs it must be dimensionally homogeneous. By equating, therefore, the dimensions of the two sides we obtain the required powers α, β and γ. Thus, writing (5.2) in dimensional form

$$\mathrm{T} = \mathrm{M}^{\alpha}\mathrm{L}^{\beta}(\mathrm{LT}^{-2})^{\gamma}, \quad \ldots \ldots \quad (5.3)$$

and equating indices of the dimensional quantities on both sides, we find

$$\begin{aligned} &\text{Mass} && 0 = \alpha, \\ &\text{Length} && 0 = \beta + \gamma, \quad \ldots \ldots \quad (5.4) \\ &\text{Time} && 1 = -2\gamma, \end{aligned}$$

and hence from (5.4) $\alpha = 0$, $\beta = \frac{1}{2}$ and $\gamma = -\frac{1}{2}$, so that equation (5.2) becomes

$$t = l^{\frac{1}{2}} g^{-\frac{1}{2}}. \quad \ldots \ldots \quad (5.5)$$

We note that the time of oscillation is predicted to be independent of the mass of the bob, a fact which is borne out by experiment. It is clear that the process we have used, which is that of **dimensional analysis**, gives us no information about **dimensionless quantities or pure numbers** such as 2π. We therefore write (5.5) in the form

$$t = k\sqrt{\frac{l}{g}}, \quad \ldots \ldots \quad (5.6)$$

where k may represent some number. We note, however, that k may be determined by a *single experiment*, on substituting the measured value of t for a given length l and a known value of g.

Another use of dimensional analysis is the testing of equations. After a long mathematical analysis it is useful to check the final equation for dimensional homogeneity by substitution of the dimensional formulæ on both sides of the equation.

We shall come across various other applications of dimensional analysis in the course of this book, but we should warn the student here that the successful use of dimensional analysis involves a very wide knowledge and experience of Physics. Correct results depend upon the selection of the correct variables (m, l and g in our pendulum example), the choice of which is guided solely by physical analogy and intuition. Again, special methods of treatment must be used *if the*

number of variables is greater than the number of primary quantities. Thus in the example chosen, there were three variables m, l and g, and three primary quantities, M, L and T, in which these could be expressed. Consequently three equations were obtained and the powers α, β and γ could be completely determined. Finally, besides dimensionless constants like 2π, there exist **dimensional constants** such as the gravitational constant G, discussed in the next chapter. Whether or not such dimensional constants should be introduced as additional variables when deducing formulæ by dimensional methods, often requires rather subtle considerations to decide.

EXERCISES

1. Explain what is meant by dimensional analysis, illustrating your answer by examples from different branches of physics.

2. A drop of liquid is suspended in another liquid of the same density but with which it is immiscible. If the drop is distorted from the spherical shape and then released, deduce by dimensional methods a formula for its period of oscillation, given that the latter depends on surface tension, density and drop radius. [$t = k\rho^{1/2}r^{3/2}T^{-1/2}$.]

3. Show by dimensional methods that the viscous retarding force on a sphere moving with slow uniform velocity through a viscous liquid is proportional to the velocity, the radius of the sphere and the coefficient of viscosity of the liquid.

CHAPTER VI

The Acceleration due to Gravity and the Gravitational Constant

The two quantities associated with the earth's gravitational field which are of particular importance are g, the acceleration due to gravity at the earth's surface and $\mathbf{G}$, the Newtonian constant of gravitation.

1. Methods of determining g.

Any motion which involves the earth's gravitation field may be used to obtain a value of g, and several possible methods are tabulated below. It is important, however, that the student should realize that a large number of them are simply laboratory exercises and that in practice the only method which is used in an accurate determination of g is the one employing the compound pendulum.

(1) *Inclined plane.*
(2) *Body rolling in a concave mirror.*
(3) *Atwood's machine.*
(4) *Simple pendulum.*
(5) *Conical pendulum.*
(6) *Bifilar suspension.*
(7) *Compound or rigid pendulum.*
(8) *Vertical vibrations of a body on a spiral spring.*

The direct determination of g by observation on the free vertical fall of a body is necessarily extremely inaccurate, since over short distances the time of fall is too short, while over large distances, owing to the high velocity acquired, the determination of the instant that the body passes a fixed point becomes very uncertain. The other methods, therefore, are methods which allow of the "dilution" of gravity. For example, on an inclined plane the acceleration of a particle becomes $g \sin\theta$ down the plane, so that by making θ small the effective acceleration may be made small. In practice, a limit is set by the rising relative magnitude of frictional forces as θ is made small. This difficulty is almost eliminated by the use of a **pendulum,** which

may be regarded as the motion of a body on a frictionless inclined plane of continually varying angle.

2. Body Rolling Down an Inclined Plane.

Fig. 1 shows two positions P and Q of a body rolling down an inclined plane. By the principle of the conservation of energy, if we assume that no energy is lost through friction, the gain in the kinetic energy (partly translational and partly rotational) of the body at Q, if it starts from rest at P, is equal to its loss in potential energy. Hence

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 = mgh, \quad . \quad . \quad . \quad . \quad . \quad (6.1)$$

where I is the moment of inertia of the body about its axis, ω is its angular velocity of rotation about that axis, m is the mass of the body, v its linear velocity, h is the vertical distance fallen, and g is

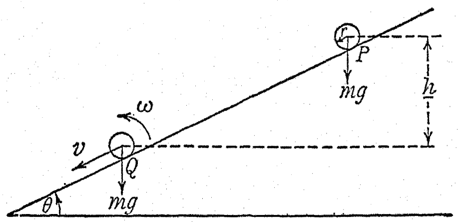


Fig. 1

the acceleration due to gravity. If mk^2 is written for I, where k is the radius of gyration about the axis of rotation through the C.G., (6.1) becomes

$$g = \frac{v^2 + k^2\omega^2}{2h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.2)$$

If r is the radius of the body, $\omega = v/r$, also, from (2.16),

$$v^2 = 2ah \operatorname{cosec} \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

where a is the acceleration down the plane.

Hence $$g = a\left(1 + \frac{k^2}{r^2}\right) \operatorname{cosec} \theta. \quad . \quad . \quad . \quad . \quad . \quad (6.4)$$

If θ is small, a is correspondingly small, and by using a stop-watch and scale the distance-time curve may be obtained. The equation of this curve is $s = \frac{1}{2}at^2$, and a may be found by substituting any corresponding values of s and t in this equation; a better result is obtained from the average of several values of a, as so found. The radius r is

obtained by direct measurement of the diameter, and k^2 can be shown mathematically to have the following values:

(1) *Solid sphere*, $k^2 = \frac{2}{5}r^2$.

(2) *Solid cylinder* or *disk*, $k^2 = \frac{1}{2}r^2$.

(3) *Hollow cylinder* or *hoop*, $k^2 = r^2$.

For a body rolling on a concave mirror we have the inclined plane replaced by an inclined plane of varying angle. It is left as an exercise to the student to show that for a sphere of radius r, oscillating to and fro in a concave bowl of radius R, the time of oscillation is given by

$$\tau = 2\pi\sqrt{\frac{7(R-r)}{5g}}. \qquad (6.5)$$

3. Atwood's Machine.

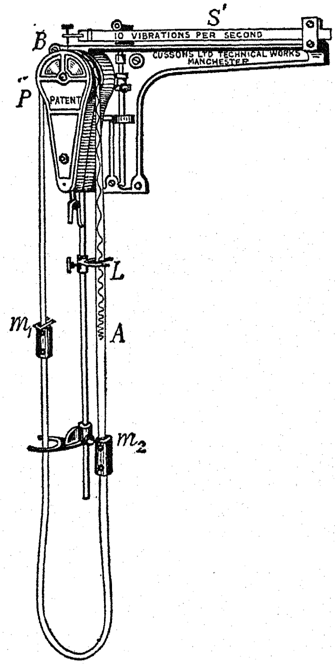


Fig. 2

In fig. 2 is shown the arrangement known as *Atwood's machine*. Two unequal masses m_1 and m_2 are joined by an endless ribbon A passing over a frictionless pulley P. When m_2 is released by removing the platform L, assuming m_2 greater than m_1, the former will commence to move down and the latter up with a uniform acceleration a. Suppose the vertical distance traversed is h, and the velocity then acquired by the masses is v. Then if the moment of inertia of the pulley wheel is I and its radius is r, we have, from the conservation of energy,

$$\tfrac{1}{2}I\omega^2 + \tfrac{1}{2}(m_1 + m_2)v^2 = (m_2 - m_1)gh, \quad . \quad . \quad (6.6)$$

where $\omega = v/r$ = angular velocity of rotation of the pulley. Hence

$$g = \frac{v^2(I/r^2 + m_1 + m_2)}{2(m_2 - m_1)h}. \quad . \quad . \quad . \quad . \quad . \quad (6.7)$$

Now if the actual acceleration of the system is a, $v^2 = 2ah$.

Hence

$$g = \frac{a(\mathrm{I}/r^2 + m_1 + m_2)}{(m_2 - m_1)}. \quad . \quad . \quad . \quad . \quad (6.8)$$

It is usual to eliminate I by carrying out the experiment with two

other masses m_2' and m_1', and observing the new acceleration a'. If this is done, we have

$$g = \frac{(m_1 + m_2) - (m_1' + m_2')}{\dfrac{(m_2 - m_1)}{a} - \dfrac{(m_2' - m_1')}{a'}}. \quad . \quad . \quad . \quad (6.9)$$

To avoid error due to friction, the machine is loaded on one side with a mass w so that it will run with *uniform velocity* when given a slight movement. This occurs if the additional weight is just sufficient to overcome the friction; the absence of acceleration indicates the absence of force and consequently the frictional force has been compensated. The weight w is kept in position throughout the determination of g and does not of course form part of the masses m_1, m_2, m_1' or m_2'. To determine the acceleration, a flat vibrating strip S′, the period of vibration τ of which has been timed by a stop-clock in a subsidiary experiment, carries a style B which marks a wavy im-

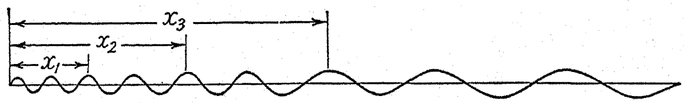


Fig. 2*a*

pression on the ribbon A. Now if x_1, x_2 and x_3 (fig. 2(*a*)) are the distances covered from rest in successive periods τ_0, $2\tau_0$, $3\tau_0$, we have:

$$x_1 = \tfrac{1}{2}a\tau_0^2,$$

$$x_2 = u\tau_0 + \tfrac{1}{2}a\tau_0^2 = \tfrac{3}{2}a\tau_0^2, \text{ since } u = a\tau_0,$$

$$x_3 = \frac{5}{2}\,a\tau_0^2,$$

so that $(x_2 - x_1) = (x_3 - x_2) = a\tau_0^2$, whence, τ_0 being known and the distances having been measured, a can be calculated.

4. Simple Pendulum.

The theory of this has already been given in Chap. III, § 9; the period, (3.7), is

$$\tau = 2\pi\sqrt{\frac{l}{g}}.$$

In practice, τ may be obtained accurately with a stop-clock; the error in g arises from the interpretation of l and from the ideal conditions assumed in the derivation of the formula. The distance l is usually taken to be the distance from the point of suspension to the centre

of the suspended spherical bob. This would be fairly satisfactory if the bob and string moved rigidly; actually at the end of a swing the inertia of the bob will cause it to take up some such position as that shown in fig. 3. Again, the mass of the string has been neglected; in fact, string and bob really act as a compound pendulum of ill-determined rigidity.

In the **conical pendulum** (fig. 3*a*), the bob is made to describe a horizontal circle, instead of vibrating in a vertical plane. The forces

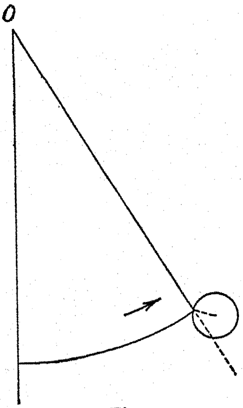


Fig. 3

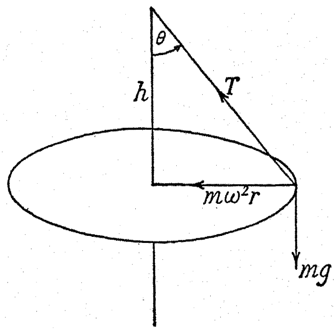


Fig. 3*a*

acting on the bob are T along the string, and mg down. Their resultant balances the centrifugal face outwards, $m\omega^2 r$. Hence, by resolving, $m\omega^2 r = T \sin\theta$, and $mg = T\cos\theta$. Thus, $\omega^2 r/g = \tan\theta = r/h$; and $\omega = \sqrt{g/h}$, $\tau = 2\pi/\sqrt{g/h}$.

5. Bifilar Suspension.

If a body is suspended by two strings the arrangement is called a *bifilar suspension*. In practice the strings are usually of equal length; when the suspended system is given a slight rotation in a horizontal plane forces are called into play which result in simple harmonic oscillations of the suspended body.

Consider the horizontal rod AB of fig. 4 supported symmetrically by two strings of length l. Let the distance between the two strings be $2b_1$ and $2b_2$ at the top and bottom respectively, and let the mass of the rod be m. When the rod is rotated through a small angle θ in a horizontal plane the rod takes up the position A′B′. Then, from the figure, if T is the tension in the strings before the bar is displaced, for vertical equilibrium

$$2T\cos\phi = mg, \qquad \qquad (6.10)$$

where $\cos\phi = (l^2 - a^2)^{\frac{1}{2}}/l$ and $a = (b_2 - b_1)$.

From the triangle AA′M

$$A'M^2 = a^2 + b_2{}^2\theta^2 - 2ab_2\theta\cos\left(90° - \frac{\theta}{2}\right);$$

therefore, since θ is small, we have approximately

$$A'M = a,$$

so that the $\angle A'PM = \phi$, and the restoring force along A′M in the plane of rotation is $T\sin\phi = Ta/l$. Since there is an equal and opposite restoring force at B′, the restoring couple is given by

$$\frac{2Ta}{l}\,.\,\sin\alpha\,.\,b_1 = \frac{2Ta}{l}\,.\,\frac{b_2\theta}{a}\,.\,b_1 = \frac{2Tb_2\theta}{l}\,.\,b_1, \quad . \quad (6.11)$$

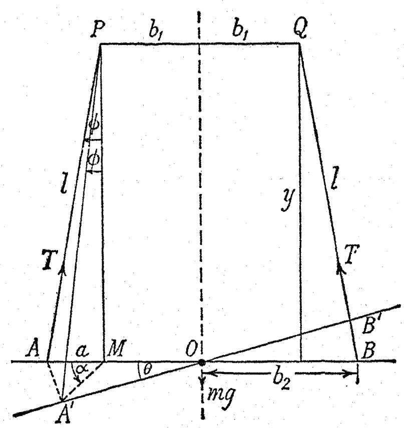


Fig. 4

where θ has been written for $\sin\theta$ and the sine formula has been applied to the triangle A′MO. From (6.10) and (6.11), the restoring couple becomes

$$mg\frac{l}{(l^2 - a^2)^{\frac{1}{2}}}\cdot\frac{b_1b_2\theta}{l}. \quad . \quad . \quad . \quad . \quad (6.12)$$

Hence, if I is the moment of inertia of the bar about a vertical axis through its centre of gravity, the equation of motion is

$$\mathbf{I}\frac{d^2\theta}{dt^2} = -\frac{mgb_1b_2}{(l^2 - a^2)^{\frac{1}{2}}}\theta. \quad . \quad . \quad . \quad . \quad (6.13)$$

If we denote the vertical distance of the bar from the points of suspension by y, $y = (l^2 - a^2)^{\frac{1}{2}}$, so that (6.13) may be written

$$\mathbf{I}\frac{d^2\theta}{dt^2} = -\frac{mgb_1b_2\theta}{y}. \quad . \quad . \quad . \quad . \quad (6.14)$$

Equation (6.14) is of the well-known type of equation (4.22), and the solution for the time of oscillation is

$$\tau = 2\pi\sqrt{\frac{Iy}{mgb_1b_2}}. \quad . \quad . \quad . \quad . \quad . \quad (6.15)$$

For the special case where the strings are vertical, $y = l$ and $b_1 = b_2 = b$, hence

$$\tau = \frac{2\pi}{b}\sqrt{\frac{Il}{mg}}. \quad . \quad . \quad . \quad . \quad . \quad (6.16)$$

Since the moment of inertia of a uniform rod can be calculated and its mass easily found, the acceleration due to gravity may be determined with the bifilar suspension. It suffers from the same disadvantages as the simple pendulum in that the mass of the strings has been neglected and that the whole constitutes a vibrating system of ill-determined rigidity.

6. Compound or Rigid Pendulum.

The accurate determination of g is carried out with a rigid pendulum, the theory of which has been given in Chap. IV, § 5. The type of pendulum commonly used is known as Kater's reversible pendulum and is shown in fig. 5. The pendulum consists of a uniform rod carrying a heavy bob at one end and two pairs of fixed knife-edges K_1 and K_2. An adjustable weight w is also provided, and oscillations are timed with the pendulum suspended first from K_1 and then from K_2. When this is done it is found that for a certain position of w, the period of oscillation τ is the same whether the pendulum is suspended from K_1 or K_2. In this position, the distance l between the knife-edges K_1 and K_2 is, as will be shown, exactly equal to the length of an ideal simple pendulum with the same period as the given rigid pendulum so that

$$t = 2\pi\sqrt{\frac{l}{g}},$$

and, t and l being known, g is obtained.

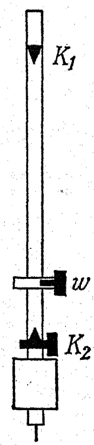


Fig. 5

To prove that the distance between K_1 and K_2 is equal to the length of the **equivalent simple pendulum,** let the centre of gravity of the rigid pendulum, which lies between the knife-edges, be at distances l_1 and l_2 from K_1 and K_2 respectively. Then, by applying equation (4.23) to the two positions, we find

$$\tau = 2\pi\sqrt{\frac{I_1}{mgl_1}} = 2\pi\sqrt{\frac{I_2}{mgl_2}}, \quad . \quad . \quad . \quad (6.17)$$

where I_1 and I_2 are the moments of inertia of the rigid pendulum about the two axes respectively. Now by a well-known theorem, known as the theorem of parallel axes, the moment of inertia of a system about any axis is equal to that about its centre of gravity plus the product mr^2, where m is the mass of the body and r the distance from its centre of gravity to the axis in question. Hence if mk^2 is the moment of inertia of the rigid pendulum about its centre of gravity, equation (6.17) may be written

$$\tau = 2\pi\sqrt{\frac{m(k^2 + l_1^2)}{mgl_1}} = 2\pi\sqrt{\frac{m(k^2 + l_2^2)}{mgl_2}}, \quad . \quad . \quad . \quad (6.18)$$

whence $$k^2 = l_1 l_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.19)$$

Resubstitution in (6.18) from (6.19) gives

$$\tau = 2\pi\sqrt{\frac{l_1 + l_2}{g}}. \quad . \quad . \quad . \quad . \quad . \quad (6.20)$$

But, for a simple pendulum of length l, the period is $2\pi\sqrt{(l/g)}$. Hence **the length of the equivalent simple pendulum is $l_1 + l_2$, the distance between the knife-edges.**

In practice, the distance between the knife-edges is determined with a scale and a travelling microscope. The period τ may be determined with a stop-clock, but the most accurate values are obtained by the method of coincidences, as explained in text-books of practical physics. It must be realized that equation (6.20) refers to ideal conditions. The observed period τ' will be connected with τ by an equation of the form

$$\tau = \tau'\{1 - (\alpha + \beta + \gamma + \delta + \epsilon + \sigma)\}, \quad . \quad . \quad (6.21)$$

where α, β, γ, δ, ϵ and σ represent small corrections.

(i) α arises from the fact that the theoretical equation (6.20) holds only for *infinitely small oscillations*. If the actual angular amplitude of the pendulum falls from θ_1 to θ_2 over the period of observation, $\alpha = \theta_1\theta_2/16$ approximately.

(ii) β is a correction to allow for the *buoyancy* due to the air surrounding the pendulum; this reduces its effective weight according to Archimedes' Principle, discussed in Chap. IX.

(iii) γ is to allow for the *kinetic energy imparted to the air* as the pendulum vibrates.

(iv) δ is a term to take into account the *friction of the air*, which tends to stop the pendulum vibrating.

(v) ϵ is a correction for *temperature rise*, since this would produce an increase in length of the pendulum, with consequent lowering of the centre of gravity and increase in the period of vibration.

(vi) σ is a correction to allow for possible *yielding of the support* on which the knife-edges are swinging. This support is generally an agate plane, and if it is screwed rigidly to a heavy beam the correction is small.

(vii) If the *knife-edges are not perfectly sharp* but are rounded cylinders a further correction is required. However, as Bessel first showed, if both pairs of knife-edges K_1 and K_2 are cylinders with the same radii of curvature the error is automatically eliminated.

The last method of determining g mentioned on p. 53 is discussed later on p. 90.

The measurement of g at sea formerly involved special and inaccurate methods, but the introduction of the submarine with its comparatively steady motion allows the use of the rigid pendulum with an accuracy approaching that obtained on land.

7. Newton's Law of Gravitational Attraction.

We shall now consider the nature of the **gravitational field,** and shall show that there is a close relation between G, the so-called Newtonian constant of gravitation, and g, the acceleration due to gravity at the earth's surface. We have already stated in Chap. II, § 8, that it was Kepler who first enunciated the laws of planetary motion. It was Newton who showed mathematically that the motion of the moon and planets, and the great bulk of the phenomena of celestial mechanics, could be accounted for if it was assumed that a force existed between two bodies of mass m_1 and m_2, inversely proportional to the square of their distance apart. **Newton's law of gravitational attraction** may therefore be written

$$\mathbf{F} = \mathbf{G}\frac{m_1 m_2}{d^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.22)$$

where F is the gravitational attractive force, and G is a universal constant termed the **Newtonian constant of gravitation.** Now a circle is a special or *degenerate* case of the ellipse, and we shall show that for the simple case of a circular orbit, Newton's law accounts for Kepler's third law. If we represent the mass of the earth by E and that of the sun by S, the gravitational force of attraction on the earth revolving in a circular orbit of radius d about the sun as centre is by Newton's law

$$F = G\frac{ES}{d^2}.$$

Since this gravitational force is equal to the product of the mass and the acceleration towards the centre, or in other words, balances the centrifugal force outwards, we have

$$E\frac{v^2}{d} = G\frac{ES}{d^2},$$

or
$$v^2 = \frac{GS}{d}$$

where v is the velocity of the earth in its orbit. Now the time of revolution is

$$t = \frac{2\pi d}{v},$$

so that
$$t^2 = \frac{4\pi^2 d^3}{GS},$$

or the square of the time of revolution is proportional to the cube of the distance, as required by Kepler's third law.

It should be noted that the interpretation of d requires careful consideration. For two solid attracting spheres, d is the distance between the centres of the two spheres; but for bodies of less regular shape, difficult integrations have sometimes to be evaluated. To show that a sphere, so far as gravitational action is concerned, behaves as though all its mass were concentrated at the centre, requires rather lengthy mathematical treatment. We shall content ourselves with observing that it is usual to prove the statement first for a thin spherical shell and then to consider a solid sphere as composed of a series of concentric spherical shells. It is important to note that it can also be shown that, for the inverse square law of attraction, a body placed *inside* a hollow gravitating sphere experiences no force. If the sphere is solid, and the body is inserted at a certain depth, the only resultant gravitational force on the body is therefore that due to the smaller sphere, concentric with the original sphere and having a radius given by the distance of the body from the centre of the sphere. This principle is involved in the mine experiments for finding the gravitational constant.

We may apply Newton's law of gravitation to find the force of attraction on a body of mass m at the surface of the earth, supposed spherical. Since, as stated above, the whole mass of the earth may be supposed to be concentrated at its centre, therefore

$$F = G\frac{Em}{R^2}, \quad \ldots\ldots\ldots \quad (6.23)$$

where E is the mass of the earth and R is its radius. Now the attractive force of the earth is what gives rise to the weight of the body, and is therefore responsible for producing the acceleration g in the body if it is free to move. Hence

$$F = mg, \quad \ldots\ldots\ldots \quad (6.24)$$

and from (6.23) and (6.24)

$$mg = G\frac{Em}{R^2},$$

or

$$g = \frac{\mathrm{GE}}{\mathrm{R}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.25)$$

Since g may be determined by experiment with the rigid pendulum and R may be found by trigonometrical survey, it follows that if G is known the mass E of the earth may be found. Finally, if ρ is the mean density of the materials of which the earth is composed, and the earth is treated approximately as a sphere,

$$\mathrm{E} = \frac{4}{3}\pi\mathrm{R}^3\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.26)$$

Three important quantities, G, E and ρ are therefore related by equations (6.25) and (6.26), and the determination of any one of these allows the remaining two to be calculated.

The force of gravitational attraction with masses of ordinary size is extremely small. For example, two equal spheres weighing together a million grammes exert a gravitational force of attraction of little over one dyne on each other when they are situated one metre apart. In order to determine G, therefore, either extremely large masses such as that of the earth itself must be used, or alternatively very sensitive balances must be used to measure the gravitational forces exerted by ordinary laboratory masses. Both methods have been used, the former now being only of historical interest. Discussing these briefly first we have:

(a) Bouguer's Mountain Experiment.

The basic idea of this experiment, which was first carried out by Bouguer on Mount Chimborazo in Peru in 1749, was to observe the deflection of a plumb-line from the vertical due to the side-ways gravitational attraction of a large mountain. If the volume of the mountain is V and its mean density ρ, then the force of attraction on the bob of the plumb-line is $GV\rho m/d^2$, where d is a rather ill-defined distance, taken approximately as that between the centre of the mountain mass and the pendulum bob (of mass m). The equilibrium of the pendulum is illustrated in fig. 6, whence

$$\tan\theta = \frac{G \,.\, V\rho m}{d^2} \times \frac{R^2}{G \,.\, Em} = \frac{V\rho}{E}\frac{R^2}{d^2}.$$

It is clear that neither V nor d can be determined with any accuracy, but the method was the first to give some idea of the magnitude of E.

The value of θ was determined from the different directions of the plumb-line with respect to that of a fixed star when its distance from the mountain was d, and when it was very great.

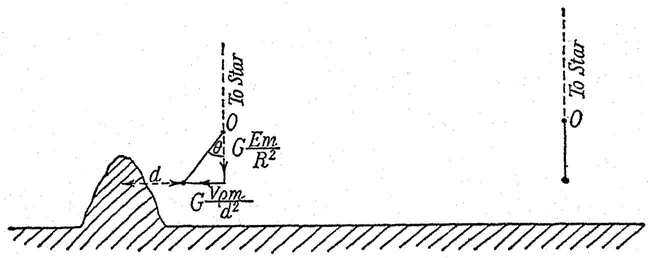


Fig. 6

(a) **Airy's Mine Experiment.**

In this experiment the times of oscillation of a pendulum at the top and bottom of a mine shaft of known depth were measured and found to be different. This difference is caused by the different accelerations due to gravity in the two positions, since, from (6.25),

$$g_1 = \frac{GE}{R^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.27)$$

and

$$g_2 = G\frac{(E - M)}{(R - h)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.28)$$

where g_1 and g_2 are the accelerations at the earth's surface and at a depth h respectively, and M is the mass of the shell of the earth of thickness h. The value of M can be obtained from the expression $M = \frac{4}{3}\pi\{R^3 - (R - h)^3\}\rho$, the value of ρ being estimated from samples of rock taken from various levels down to a depth h. Hence, to a rough approximation, estimates of G and E may be obtained by combining (6.27) and (6.28).

All the laboratory methods depend upon the use of the torsion balance for estimating the effects of gravitational forces between bodies of laboratory dimensions.

(c) **Boys' Apparatus.**

The experiment was first carried out by Cavendish in 1798; the principle lay in fixing two small spheres to the ends of a rigid horizontal rod which was suspended by a thin metal torsion wire. Two large lead spheres were then placed on opposite sides of the suspended rod and spheres, and the gravitational forces produced a measurable twist in the suspension. The main difficulty in the original experiment of Cavendish lay in the avoidance of convection air currents, for the torsion beam was some 6 feet in length. By introducing the

use of quartz fibres for the torsional suspension, Boys was able to reduce the size of the beam without loss of sensitivity. A form of Boys' apparatus is shown in fig. 7. A central vertical quartz fibre T is suspended from a metal disk H termed the torsion head. The fibre carries a horizontal glass beam; from grooves in both ends, other quartz fibres of unequal length hang vertically, each supporting a small gold sphere S. This suspension system is hung inside a glass tube, of internal diameter about 3·8 cm., and is thus protected from draughts. Outside the tube, two equal spheres of lead about 11 cm. in diameter are suspended at equal distances from the axis. The centres of these spheres are respectively situated in the same horizontal planes as the centres of the gold balls so that each lead sphere exerts a large attraction only on the gold sphere to which it is closest. The angular deflection θ is obtained by using a lamp, mirror, scale and telescope, the mirror being the horizontal glass beam itself. The large spheres are then swung to the reverse side of the gold spheres and the maximum deflection in the opposite direction is obtained. If the couple required to produce unit angular deflection is C, then we have

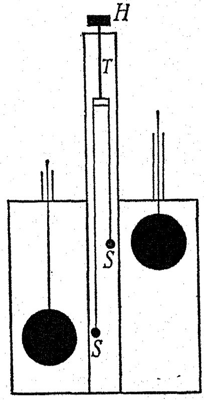


Fig. 7

$$C\theta = \frac{G \, . \, mMl}{d^2}, \qquad \ldots \ldots \qquad (6.29)$$

where m and M are the masses of the gold and the lead spheres respectively, d the distance between the centres of two spheres attracting each other, in the final position of equilibrium, and l is the distance between the centres of the gold spheres, which is approximately the length of the torsion beam. The quantity C is found from the time of oscillation τ of the suspended system, since, if its moment of inertia (a quantity which is calculated for this symmetrical arrangement) is I, then

$$\tau = 2\pi\sqrt{\frac{I}{C}}. \qquad \ldots \ldots \qquad (6.30)$$

The most accurate value of G is probably that obtained by Heyl in 1930. A torsion balance was again used, but details of the method are beyond the scope of this book.

The dimensional formula for G is obtained from its definition in equation (6.22) and found to be $M^{-1}L^{3}T^{-2}$. Its value is

$$6{\cdot}67 \times 10^{-8}\ \text{cm.}^3\ \text{gm.}^{-1}\ \text{sec.}^{-2},$$

and hence the mean density of the earth is found to be about 5·5. Since the density of the surface crust is much less than this, the interior of the earth must consist of materials of high density.

8. Variations of *g*.

Equation (6.25) was deduced on the assumption that the earth is a perfect sphere, which it is not. The earth is more nearly a spheroid, being flattened at the poles and bulging at the equator. Hence the value of g varies from place to place on the earth's surface. We should expect it to be greater at the poles than at the equator. This is found to be so, although another cause, which we shall discuss in a moment, assists in making this difference greater than it would otherwise be.

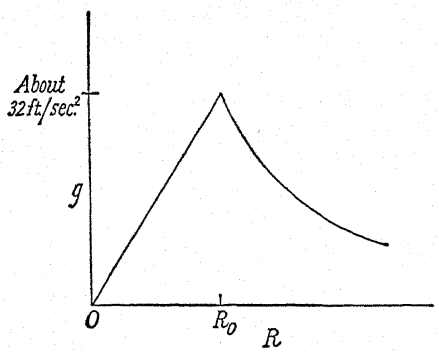


Fig. 8

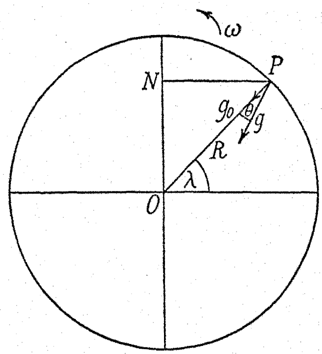


Fig. 9

We may note here that although from equation (6.25) g varies inversely as the square of the distance from the centre of the earth (regarded as a sphere) for points outwards from the earth's surface, yet, for points in the interior of the earth, g varies directly as the first power of the distance. This behaviour, which is illustrated in fig. 8, may be explained from equations (6.25) and (6.26), for combination of these gives

$$g = \frac{4}{3}\mathrm{G}\pi\mathrm{R}\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.31)$$

so that for $R < R_0$, the radius of the earth, the acceleration due to gravity is proportionally less.

We must next consider the effect of the centrifugal force, arising from the rotation of the earth, on the apparent value of the acceleration due to gravity at different points on the earth's surface. Treating the earth once more as a sphere, consider with reference to fig. 9 the forces acting on a mass m situated at a point P on the earth's surface. Representing the uniform angular velocity of the earth by ω, the true

acceleration due to gravity by g_0, the observed value by g and the angle of inclination between the apparent and true vertical by θ, if we resolve the forces parallel and perpendicular to OP, we have, since mg is the resultant of mg_0 and $m\omega^2$. PN outwards,

$$mg\cos\theta = mg_0 - mR\omega^2\cos^2\lambda, \quad . \quad . \quad . \quad (6.32)$$

and

$$mg\sin\theta = mR\omega^2\cos\lambda\sin\lambda, \quad . \quad . \quad . \quad . \quad (6.33)$$

where λ is the latitude of P. Hence

$$\tan\theta = \frac{\omega^2 R\cos\lambda\sin\lambda}{g_0 - \omega^2 R\cos^2\lambda}, \quad . \quad . \quad . \quad . \quad (6.34)$$

or, eliminating θ,

$$g = g_0\left(1 - \frac{\omega^2 R\cos^2\lambda}{g_0}\right), \quad . \quad . \quad . \quad (6.35)$$

where we have expanded by the binomial theorem and neglected higher powers of $\omega^2 R/g_0$, which equals 1/289; otherwise: since θ is small, we may put $\theta = 0$ in (6.32). Substitution of the appropriate values in equation (6.35) shows that the acceleration due to gravity will change by about 0·3 per cent, owing to centrifugal force alone, as we proceed from the poles to the equator. Local variations in the magnitude and direction of g will also occur, owing to irregularities in gravitational attraction such as that caused by the mountain in Bouguer's experiment.

9. Variations in G.

A great many experiments have been performed to test whether G is a universal constant or whether, for example, the interposition of different bodies between two gravitationally attracting masses causes the force of attraction to differ from that to be expected from the simple relation $F = Gm_1m_2/d^2$. We shall see in Part 5 that the force between two magnetic poles and between two electric charges is also that of the inverse square. For these forces, the magnitude of the force depends very much on the nature of the medium separating the centres of force. In the magnetic case, this is said to be due to the **permeability** of the intervening medium. We may state immediately that there is no experimental evidence for what might be termed gravitational permeability. All attempts to detect variation in G due to different materials, temperature change, orientation of crystallographic axes, and so on, have so far produced negative results.

EXERCISES

1. Enumerate methods for determining the acceleration due to gravity and discuss the accuracy obtainable by different methods

2. Give an account of an accurate method for finding the acceleration due to gravity and point out the corrections to be applied if a very accurate value is required.

3. Write a short essay on Weighing the Earth.

4. Describe in detail Boys' method for finding the Newtonian constant of gravitation.

5. Explain clearly why the acceleration due to gravity is less at the equator than at the poles and deduce a general formula for the acceleration due to gravity in any latitude in terms of its value for a spherical non-rotating earth.

6. Find the acceleration of any solid sphere, rolling down a plane inclined at 20° to the horizontal. [$0{\cdot}24g$.]

7. Determine the acceleration of masses of 200 and 210 gm. respectively attached to an Atwood's machine, neglecting friction and treating the pulley as a disk of mass 100 gm. [$g/46$.]

8. What is the length of the equivalent simple pendulum for a pendulum which consists of a sphere of radius 10 cm. suspended by a light string 50 cm. long? [60·67 cm.]

9. Find the time of oscillation of a cylindrical rod of length 100 cm. and radius 1 cm., suspended horizontally by two vertical strings of length 50 cm., attached to points on the rod 20 cm. from each end, the rod vibrating in a horizontal plane. [0·683 sec.]

10. Prove that the gravitational force acting on a particle inside a closed spherical mass-shell is zero (Gauss' theorem). Hence show that the shell behaves for external points as though all its mass were concentrated at the centre.

11. Using the results of Q. 10, show that a solid sphere, considered as a number of concentric spherical shells, exerts a gravitational force at external points as though all its mass were concentrated at the centre.

12. If a pendulum is released from rest with its centre of gravity in the same horizontal line as the axis of rotation, determine at what angle the pendulum is inclined to the vertical when the horizontal thrust on the axis is a maximum. [45°.]

CHAPTER VII

Gyroscopic Motion*

1. Conservation of Angular Momentum.

Consider the experiment illustrated in fig. 1. A man is seated in a chair which is free to revolve about a vertical axis. He holds a bicycle wheel which he supports by a short axle held in the hands. The wheel is given a spin by an outside observer, and the man then moves the axle of the rotating wheel in a sideways direction so that the plane of the wheel becomes vertical. He finds that it requires considerable strength to do this, and he also finds that his chair commences to rotate about its vertical axis. If he moves the wheel back again he finds that his chair rotates back again to its original position. The explanation of these phenomena is the conservation of angular momentum, mentioned briefly in Chap. IV, § 4. We have to consider the system of man, chair and wheel as a whole. Initially the system has a certain angular momentum about the axis of the chair. When the man turns the axle of the wheel so that it points horizontally, the wheel acquires angular momentum about a horizontal axis; this comes ultimately from an external couple applied at the bearings of the axis of the chair. But these bearings cannot communicate angular momentum about the axis of the chair itself, and therefore the angular momentum of the system of man, chair and wheel, about that axis, must retain its original value. It follows that the man and chair must rotate.

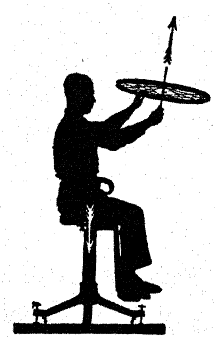


Fig. 1

If the moment of inertia of the rotating wheel about the vertical axis of the chair is I_1 and its original angular velocity of movement about this axis was ω_1, then for the conservation of angular momentum about the axis of the chair,

$$I_1\omega_1 = I_2\omega_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.1)$$

where I_2 is the moment of inertia of the whole system about the axis of the chair and ω_2 is the final common angular velocity of rotation.

Since I_2 is much greater than I_1, the angular velocity of rotation of man and chair is correspondingly smaller than the original angular velocity of the wheel.

The conservation of angular momentum explains the ability of a cat always to land on its feet when it is dropped. The cat achieves this in the following manner. Suppose it is released lying on its back. It immediately swings its legs, head and tail sideways; this results in an oppositely directed rotation of the remainder of its body. Since the latter has greater moment of inertia than the former its angular velocity of rotation is not very large, but it is sufficient to produce a small rotation before the legs and tail have reached the end of their sideways swing. If now the cat were to swing its legs and tail back by the same route as it originally took, its body would rotate back again and the cat would once more be on its back. To avoid this the cat draws in its extremities *radially*. This process introduces no angular momentum and hence the body remains turned through a small angle. The cat then throws its legs sideways again in such a direction as to increase further the rotation of the body and then again draws the legs in radially. Proceeding in this way with sufficient rapidity the cat can rotate itself so as to be feet downwards for landing, for any depth of drop greater than a few feet. The whole process can be watched in detail by means of the slow motion camera. The student is now in a position to see why an acrobat can rotate so rapidly in a curled position, how a diver can perform double somersaults before reaching the water, and how a ballet dancer is able to pirouette so rapidly upon her toes. In all instances a certain angular momentum is imparted to the body by the operator when the body is in a position such that its moment of inertia about the axis of rotation is large. The moment of inertia about that axis is then made small by radial contraction and the angular velocity increases automatically so as to keep the angular momentum constant.

2. Gyroscopic Action.

One form of the **gyroscope** or **gyrostat** consists of a heavy flat metal wheel which can rotate freely about a central axle. This axle is supported in gimbals which are themselves supported on a point, and the whole is counterpoised so that the centre of gravity of the system lies at the point of suspension as shown in fig. 2. The whole constitutes a special form of top, and the wheel can be rotated at a high speed either by a string or as in the gyrostatic compass, by an electric motor. Suppose now the axle is held and the wheel is spun, the whole then being released. The system commences with a definite

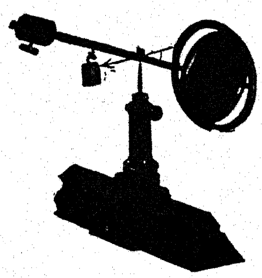


Fig. 2

angular momentum about the axis of rotation of the disk, which we shall assume to be rotating in a vertical plane, so that the axis of the disk is horizontal. Suppose a small weight w is suspended on the horizontal axis on the side opposite to the wheel. The weight applies a couple G to the system, and had the wheel been at rest the loaded side would simply have fallen and continued to fall until it met some resistance; but now that the disk is rotating what is observed is a small descent of the weight, after which it ceases to fall. At the same time the whole apparatus commences to rotate or **precess** about the vertical axis of support. The reason for this precession is as follows. We may represent the angular momentum of the disk by the vector OP $= M$, and the couple due to the weight by G. The latter tends to produce angular momentum dM in a vertical plane about a horizontal axis in a time dt. The total angular momentum of the system therefore becomes represented by OP′ in fig. 3, where PP′ $= dM$. The reason for drawing PP′ perpendicular to OP is that the angular rotation which the couple due to the weight attempts to produce is about a horizontal axis in the direction of PP′, and it can be shown that angular momenta may be regarded as vectors and represented by lines drawn of length proportional to their magnitude and in the direction of the axes about which they operate. In time dt, therefore, the axis OP has rotated through an angle $d\alpha$, given by

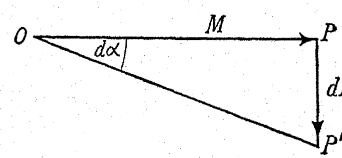


Fig. 3

$$M d\alpha = dM. \qquad (7.2)$$

Now if the angular velocity of precession is ω_p,

$$d\alpha = \omega_p dt. \qquad (7.3)$$

Also from equation (4.20) $$G = \frac{dM}{dt}. \qquad (7.4)$$

Hence, from (7.2), (7.3) and (7.4), the angular velocity of precession is given by

$$\omega_p = \frac{G}{M} = \frac{G}{I\omega}, \qquad (7.5)$$

where I is the moment of inertia of the wheel about its axis, and ω is its angular velocity about that axis. From (7.5), the greater the couple G, that is, the greater the weight w, the faster the rate of precession. On the other hand, the larger the angular velocity of the wheel the smaller the rate of precession. Owing to friction, the angular

velocity ω of the wheel gradually decreases and consequently, as time goes on, the rate of precession increases, and the dip of the weight w becomes more pronounced.

Perhaps the most important practical application of the principle of the gyroscope is the gyrostatic compass. This instrument is used in the steering of ships and aeroplanes, the magnetic compass now taking secondary place, and being used chiefly as an emergency instrument. The gyrostatic compass consists essentially of a heavy disk which is driven by an electric motor and supported freely in frictionless gimbals. If it is set rotating with its axis in the meridian, then owing to the persistence of the direction in space of the angular momentum, the axis continues to point in this direction no matter how the supporting body moves. Actually, small deviations do arise due, among other things, to the rotation of the earth, with the result that tables of corrections have to be applied to deduce the true meridian at any latitude, just as for the magnetic compass. The gyrostatic compass, however, has the advantages of being free from magnetic disturbance due to the presence of iron, of possessing higher torsional rigidity and even of being able, by suitable coupling, to operate the steering-gear of the vessel.

EXERCISES

1. Explain the precession which occurs when a small weight is attached to the horizontal axis of a gyrostat. How has gyroscopic action been applied to replace the magnetic compass?

2. Describe experiments which demonstrate that angular momentum is a vector quantity. How is a cat able to land on its feet even when it is released lying on its back?

3. A circular disk is spinning about a diameter when suddenly it is held fixed at a point on the circumference midway between the ends of the diameter. Show that the new angular velocity of rotation is $\omega/5$, where ω was the original angular velocity.

CHAPTER VIII

Elasticity

1. Introduction.

The student is now in a position to appreciate the statement that advances in the understanding of nature are achieved by making abstractions. In elucidating the laws of dynamics we have seen that the concept of the material particle and of the rigid body have been of the utmost importance. However, experiment shows that in the vast majority of instances met with in practice actual results are not in exact agreement with those to be expected from a direct application of the laws deduced hitherto. These discrepancies are not due to the theory being at fault, but to over-simplification in our treatment of actual bodies.

No body, for example, is perfectly rigid. When a large weight is attached to a long, thin, horizontal bar gripped at one end, not only are certain forces called into action, but the bar is **bent**. Usually the bending is accompanied by vertical vibrations of the bar which gradually die away. The energy of vibration cannot, of course, by the principle of conservation of energy, have been destroyed; and careful measurements would show that the temperature of the bar had been raised.

We concentrate in this chapter on **elasticity**, the deformation of bodies when subjected to forces. If a body entirely recovers its original size and shape when the deforming forces are removed, it is said to be **perfectly elastic**. If it retains completely its new size and shape it is said to be **perfectly plastic**. Actual bodies lie between those two extremes, but most bodies are nearly perfectly elastic over a certain range of force, and it is with such ranges that we shall be mainly concerned. Some substances, especially single crystals, exhibit different elastic properties in different directions. Such bodies are said to be **anisotropic**. Their treatment is complicated, and we shall only consider substances whose elastic properties do not depend on direction; these are said to be **isotropic**. Fortunately, metal rods, bars and wires, which are the most important elastic bodies occurring in practice, are approximately isotropic in their elastic behaviour.

2. Strain and Stress.

The **strain** of an elastic body may be described in general terms as the change of shape or the fractional change of size (or both) which that body undergoes. It is therefore a dimensionless quantity. The forces which produce the strain are often loosely referred to as the stresses, but the correct description of **stress is force per unit area.** A stress, therefore, has dimensions $ML^{-1}T^{-2}$. It is usual to define the stress more exactly. If, in fig. 1, the force F is acting on the plane area A at an angle to its surface, the normal component of F divided by A is termed the **normal stress,** and the tangential component of F divided by the area is termed the **tangential stress.** The plane area A may be a surface in the interior of a body, separating two portions P and Q of the body. Then P acts on Q across A with a certain force per unit area, and Q acts on P with an equal and opposite force. The force per unit area, in this double aspect, is also called the **stress across** A.

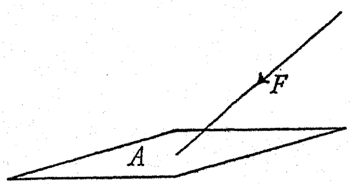


Fig. 1

With respect to stress and strain, a perfectly elastic body would exhibit the following properties:

(*a*) A given stress would always produce the same strain.

(*b*) Maintenance of a given stress would maintain the same strain.

(*c*) Removal of stress would result in complete disappearance of strain.

3. Hooke's Law.

We shall now consider the experiment shown in fig. 2. In this apparatus, due to Searle, a frame-work CC′D′D is supported by two vertical wires A, A′ fastened to clamps at F. Inside the frame-work rests a spirit-level L supported by the horizontal bar H and the end of the thick screw S. A large graduated drum-head is attached to S and moves over a vertical scale R. From one side of the frame-work is suspended a heavy constant weight M and from the other a heavy scale pan P. The spirit-level is first adjusted to the horizontal position by turning the drum-head on S. A known load is then placed in P, whereupon, since K, K′ are loosely pivoted, the increase in length of the wire A′ results in tilting of the spirit-level. The distance through which S has to be turned to bring the level back to horizontal then gives directly the increase in length of the wire A′.

In fig. 3 are shown curves which are given by such an apparatus if the load is plotted against the extension. These curves are known as the load-extension, stress-strain, or briefly *p*-*e* curves for a material. Considering (*a*) we note that for a considerable distance from the origin

the curve is a straight line, indicating that the extension is directly proportional to the load. This fact is known as **Hooke's Law.** It is observed to hold in fig. 3(a) up to the point P which is termed the **limit of proportionality.** Further increase in the load leads one to a point known as the **elastic limit.** Up to this point the wire will return to its original length once the load is removed. Beyond this point a

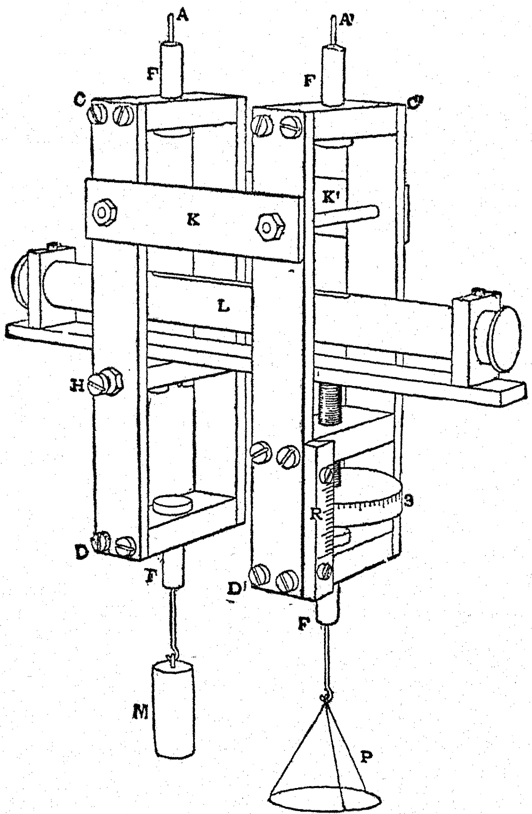


Fig. 2

permanent set is produced. The extension is observed to increase very rapidly beyond the point S for only a small increase in load. The material of the wire actually *flows*, beyond S, the so-called **yield-point,** certain sections of the wire decreasing rapidly in diameter. Fracture occurs at the point Z, the value of p_z and e_z being termed the **breaking stress** and the **breaking strain** respectively. We note that the breaking stress in fig. 3(a) is less than the maximum stress at the point B; the latter is a point of unstable equilibrium.

By using suitable apparatus the behaviour of materials under compression instead of extension may be observed. Hooke's Law is again initially obeyed, but eventually fails, and the specimen finally collapses beyond the **crushing limit** S′ (see lower part of fig. 3(*a*)).

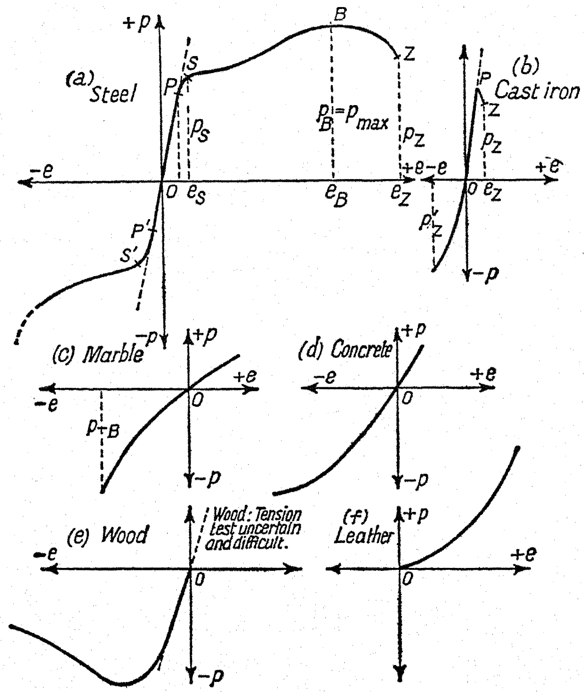


Fig. 3

Consideration of figs. 3(*b*), (*c*), (*d*), (*e*) and (*f*) will show that materials like marble and wood do not obey Hooke's Law over any appreciable range.

4. Moduli of Elasticity.

The slope of the *p*-*e* curve over the region of proportionality is characteristic of the material of the wire, and is termed **Young's modulus of elasticity** of the material. It is defined as

$$q = \frac{\textbf{stress}}{\textbf{strain}} = \frac{\textbf{Applied load per unit area of cross-section}}{\textbf{Increase in length per unit length}} = \frac{WgL}{\pi r^2 l}, \tag{8.1}$$

where W is the load, g the acceleration due to gravity, L the original

length of the wire, r its radius and l its extension (increase of length) under the load W.

If the load is expressed in grammes, and lengths in centimetres, the formula $WgL/(\pi r^2 l)$ gives Young's modulus in **dynes per sq. cm.** In Engineering, Young's modulus and other similar moduli are usually expressed in **tons per sq. in.**

There are two other moduli of elasticity in common use; these are the **rigidity modulus** and the **bulk modulus** respectively. There is also an elastic constant termed **Poisson's ratio.** These quantities are defined as follows. Consider a cube of side ABCD fixed at the base and under the action of tangential forces in the direction AA′BB′ (fig. 4). Equal tangential forces are set up along the sides, CD, CB, AD. The cube takes up the form A′B′CD, that is the volume remains unaltered; such a strain is termed a **shear,** and is measured by the angular deformation θ. As the tangential stress is increased, over a limited range the angular deformation θ increases proportionally and the *rigidity modulus* is defined by

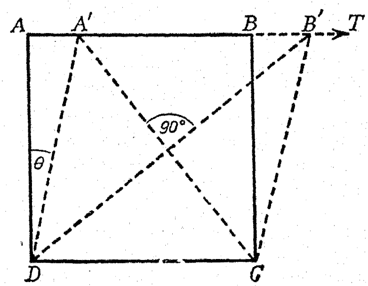


Fig. 4

$$n = \frac{\textbf{stress}}{\textbf{strain}} = \frac{\textbf{tangential force per unit area}}{\textbf{shear}\,\boldsymbol{\theta}}. \quad . \quad (8.2)$$

Finally, if an isotropic body such as a sphere is uniformly compressed simultaneously in all directions it will retain its original spherical shape but undergo a reduction in volume. Over a limited range the fraction (diminution in volume/original volume) or *volume strain* is found to be proportional to the applied force per unit area, and the *bulk modulus* is defined by

$$\mathbf{K} = \frac{\textbf{stress}}{\textbf{strain}} = \frac{\textbf{compressive or tensile force per unit area}}{\textbf{change in volume per unit volume}}. \quad (8.3)$$

Since strain has no dimensions, moduli of elasticity have the same dimensions as stress.

Returning to our experiment with the stretched wire, if careful measurements are made it is found that as the extension increases, the radius of the wire decreases. Further, the ratio (decrease in radius/original radius) is proportional to the longitudinal stress. *Poisson's ratio* is defined as

$$\sigma = \frac{\textbf{fractional change in radius}}{\textbf{fractional change in length}} = \frac{\Delta r}{r} \cdot \frac{l}{\Delta l}. \quad . \quad (8.4)$$

***5. Rigidity Modulus.**

It is not usual to attempt to measure the rigidity modulus with the cubical arrangement used in its definition. The apparatus used is shown in fig. 5, which illustrates *Barton's statical method*. The specimen, which is in the form of a circular wire, hangs vertically, being clamped at A and having a brass cylinder attached firmly to it at B by means of a set-screw. A torsional couple is applied to the wire by cords which pass round the brass cylinder and carry weights in the

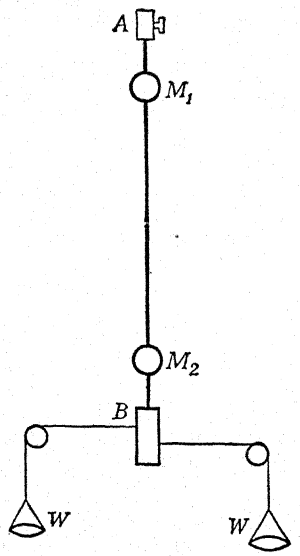


Fig. 5

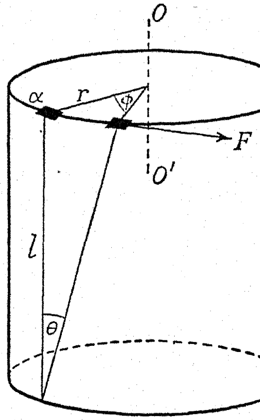


Fig. 6

scale-pans W. The angular twist in the wire between points a distance l apart is obtained by fixing two mirrors M_1 and M_2 to the wire by means of set-screws and using the usual lamp and scale method. It remains to show that the rigidity modulus of the wire, n, is given by the equation

$$n = \frac{4lWag}{\pi R^4 \varphi}, \qquad \ldots \ldots \quad (8.5)$$

where a is the radius of the brass cylinder, R is the radius of the wire, and ϕ is the angle of twist. Referring to fig. 6, consider an element, area α, of the cross-section of the wire at a distance r from its axis of symmetry. Let the wire be fixed at its lower end which is a vertical distance l from α, and let α be twisted through an angle ϕ by an ex-

ternal couple Q. Then, if the tangential stress across a is F, the element of couple which this contributes about the axis is

$$dQ = Far. \quad \quad (8.6)$$

Now the tangential stress produces a shear θ in the wire and from fig. 6

$$r\phi = l\theta. \quad \quad (8.7)$$

Also, by definition, $$n = F/\theta. \quad \quad (8.8)$$

Hence, from (8.6) and (8.8),

$$dQ = \frac{n\phi}{l} ar^2,$$

so that the total couple Q about the axis is

$$Q = \int dQ = \frac{n\phi}{l}\int_0^R 2\pi r\,dr\,r^2 = \frac{\pi n\phi R^4}{2l}, \quad \quad (8.9)$$

for the element of area a over which the same force is experienced is the circular annulus $2\pi\,rdr$. Since Q is equal to the applied couple, $2Wag$, this proves (8.5).

*6. Bulk Modulus.

We shall see in Chap. IX that liquids have the property of transmitting equally in all directions any pressure which is applied to them. In determining the bulk modulus, therefore, the body is immersed in a liquid and subjected to a measured hydrostatic pressure. The pressure is usually produced by the action of a piston on a cylinder, that

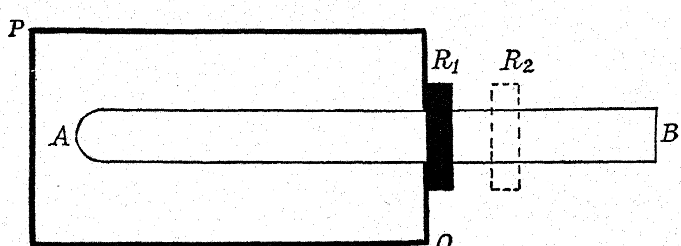


Fig. 7

is, a hydraulic ram. For many years it was possible only to use moderate pressures since at higher pressures the liquid escaped between the cylinder and the piston. In recent years a special packing device has been introduced by Bridgman which allows of pressures up to the bursting point of the container.

To find the bulk modulus of a solid an apparatus of the type shown in fig. 7 is placed inside a liquid and subjected to a high, measured

pressure. The apparatus consists of a heavy steel cylinder PQ, enclosing the specimen AB, which is in the form of a rod. The contraction of the rod relative to the cylinder is measured by the movement of a fairly loose-fitting ring R_1, which during the contraction moves to R_2, in which position it remains after the pressure is removed. Correction amounting to a few per cent has to be applied for the increase in length of the container.

The method measures the *longitudinal* strain e, due to the applied hydrostatic pressure. The change in *volume* which a sphere of the material, of original volume unity, would experience under the same pressure as that applied to the rod would be $3e$ to a first order of approximation. This follows from consideration of a cube, the side of which before compression is unity. The decrease in volume is $1 - (1 - e)^3 = 3e$, approximately, since e is small. The high pressures are measured with the *free piston gauge* introduced by Amagat. This consists of a piston which is very accurately fitted to a vertical cylinder, the latter being let into the side of the main vessel in which the compressing liquid is placed. The pressure is then measured directly from the load which must be applied to the top of the piston in order to maintain equilibrium.

We may note that, if the rigidity modulus and Young's modulus have been measured for a specimen, the bulk modulus may be calculated from (8.14).

*7. Relations between the Elastic Constants.

The elastic constants are interdependent, since any change in the size and shape of a body may be obtained by first changing the size but not the shape, as in a volume strain, and then changing the shape but not the size by means of a shear or shears. To deduce the relation, consider a cube of material as shown in fig. 8, with axes of x, y and z parallel to the edges of the cube. Let the cube be subjected to stresses X, Y and Z along these three directions and consider the strains produced. We know from our experiment on Young's modulus and Poisson's ratio that any one of these forces results in an extension of the cube along the direction in which that force acts and a contraction in directions at right angles to this. Since the extensions and contractions are proportional to the forces, if λ represents the constant of proportionality for extensions and μ that for contractions, we have

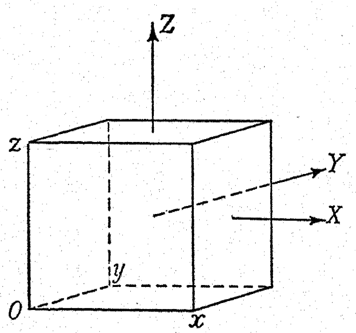


Fig. 8

for the total extensions of the cube in each of the three directions:

$$\left.\begin{aligned} e_x &= \lambda X - \mu(Y + Z), \\ e_y &= \lambda Y - \mu(X + Z), \\ e_z &= \lambda Z - \mu(X + Y). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8.10)$$

We apply (8.10) to the three moduli of elasticity. For *Young's modulus* we are concerned with only one force X, the other two being zero. Hence

$$e_x = \lambda X.$$

But Young's modulus q is, from (8.1), equal to X/e_x, so that

$$q = \frac{1}{\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.11)$$

For the *rigidity modulus*, for reasons to be given in a moment, in equations (8.16–8.18), $Y = -X$, $Z = 0$, and $n = \frac{1}{2}X/e_x$, hence

$$e_x = (\lambda + \mu)X,$$

and
$$n = \frac{1}{2(\lambda + \mu)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.12)$$

Finally, for the *bulk modulus*, since the forces are applied equally in all directions, $X = Y = Z$, and

$$e_x = (\lambda - 2\mu)X,$$

and, since the volume strain is three times the linear strain,

$$K = \tfrac{1}{3}X/e_x,$$

so that
$$\mathbf{K} = \frac{1}{3(\lambda - 2\mu)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.13)$$

Eliminating λ and μ from equations (8.11), (8.12) and (8.13), we obtain

$$q = \frac{9n\mathbf{K}}{3\mathbf{K} + n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.14)$$

Also *Poisson's ratio*, by definition, is

$$\sigma = -e_y/e_x, \text{ for the case } Y = Z = 0.$$

Hence from (8.10),
$$\sigma = \frac{\mu}{\lambda},$$

or, in terms of K and n,
$$\sigma = \frac{3\mathbf{K} - 2n}{6\mathbf{K} + 2n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.15)$$

Equation (8.15) may be written $3K(1-2\sigma)=2n(1+\sigma)$. Since K and n are both positive, $1-2\sigma$ and $1+\sigma$ must have the same sign, so that σ cannot be greater than $\frac{1}{2}$ nor less than -1; this deduction from theory is borne out by experiment.

It remains to show that the statements on which (8.12) is based are correct. We consider the cube in fig. 4 held at the base and experiencing the tangential stress X. It becomes strained through an angle θ into the figure A′B′CD, the diagonal DB increasing in length to DB′ and the diagonal AC decreasing in length to A′C. Such extensions and compressions could be produced by the action of stretching and compressing forces Q and P acting in the directions of these diagonals. These forces would have magnitude $F \sec 45° = P = Q$ if F is the actual force producing the stress X in the x-direction, for P or Q resolved in the x-direction is to be equivalent to F, as we see

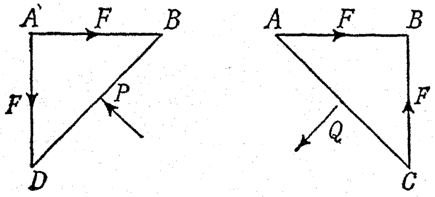


Fig. 9

by considering the equilibrium of the wedges ABD, ABC (fig. 9). Now the diagonal areas of the cube across which P and Q are acting are equal to $A \sec 45°$ where A is the top area of the cube across which the force F is acting. Hence the *diagonal stresses* produced by P and Q are proportional to $P/(A \sec 45°) = F \sec 45°/(A \sec 45°) = F/A = X$. The tangential stress X is therefore equivalent to two stresses X and $-X$ along the diagonals of the cube. If we identify the direction DB with the x-axis of the cube in fig. 8, and the direction AC with the y-axis, we obtain the first of the statements preceding (8.12), namely, $Y=-X$, $Z=0$.

To show that $n = X/2e_x$, taking the diagonal of the unstrained cube as having unit length, we have

$$\tan \mathrm{DA'O'} = \frac{\mathrm{DO'}}{\mathrm{A'O'}} = \frac{1+e_x}{1+e_y}, \quad . \quad . \quad . \quad (8.16)$$

where we shall insert the negative value of e_y later.

But $\theta = \angle \mathrm{DA'B'} - \angle \mathrm{DAB} = 2\angle \mathrm{DA'O'} - 90°$; hence

$$\tan\frac{\theta}{2} = \frac{\tan \mathrm{DA'O'} - 1}{1+\tan \mathrm{DA'O'}} = \frac{e_x - e_y}{2+e_x+e_y}. \quad . \quad . \quad (8.17)$$

Now $e_x = -e_y$, so that (8.17) becomes for small angles

$$\tan\frac{\theta}{2} = \frac{\theta}{2} = e_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.18)$$

Now, by definition, the rigidity modulus

$$n = \frac{X}{\theta} = \frac{X}{2e_x},$$

which is the other relation used in deriving (8.12).

***8. Bending of Beams.**

The bending of loaded beams is of great practical importance in structural engineering; and conversely, observation of the depression of loaded beams in the laboratory gives an accurate method for measuring elastic moduli. When a beam is bent by an applied couple, the filaments of the beam are extended on the outside of the curve and compressed on the inside. The term **neutral axis** or **neutral filament** is applied to the central filament which experiences no change in length. Suppose the rod ABCD, shown in fig. 10, is bent into a circle and that the radius of curvature of the neutral axis PQ is ρ. Considering a filament P′Q′ of the rod a distance z (= PP′) from PQ, we have

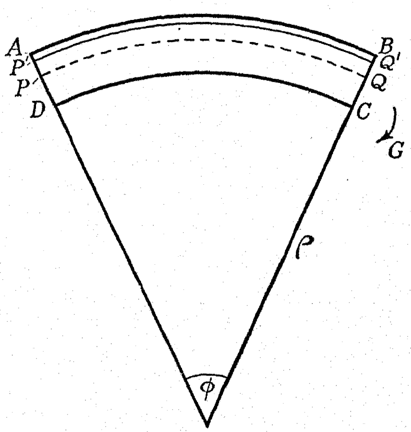


Fig. 10

$$P'Q' = (\rho + z)\phi,$$

so that the extension of this filament is

$$P'Q' - PQ = (\rho + z)\phi - \rho\phi = z\phi, \quad . \quad . \quad . \quad (8.19)$$

and the horizontal strain is $z\phi/\rho\phi = z/\rho$. If the area of cross-section of the filament is a, then if q is Young's modulus for the material of the beam,

$$q = \frac{\text{stress}}{\text{strain}} = \frac{X}{z}\rho,$$

so that the force across the area is

$$Xa = \frac{qza}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.20)$$

Consider a **cross-section** of the beam perpendicular to the neutral axis and cutting it at a point K. The line through K at right angles to the plane of the paper (the plane of bending) lies in this cross-section, and is called its *axis*. The forces such as $X\alpha$ across the cross-section combine to form a couple, the magnitude of which is found by taking the sum of the moments of the forces $X\alpha$ about the axis of the section. The contribution of a single force $X\alpha$ to the couple is $X\alpha \,.\, z = q\alpha z^2/\rho$; hence the total couple or **bending moment** exerted by the stresses in all the filaments which make up the rod is

$$G = \frac{q}{\rho}\Sigma\alpha z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.21)$$

The quantity G must be just equal to the exterior bending couple which has been applied, when the rod is in equilibrium. The expression $\Sigma\alpha z^2$ is analogous to the moment of inertia as defined in (4.18). It differs in that the element of mass dm has been replaced by an element of area. It is therefore termed the **geometrical moment of inertia** of the cross-section of the beam about its axis. Just as Mk^2 is written for a physical moment of inertia, where M is the whole mass and k is the radius of gyration, so we may write Ak^2 for the geometrical moment of inertia, where A is the surface area. Equation (8.21) is therefore generally written

$$G = \frac{qAk^2}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.22)$$

We have only dealt with the simple case when the beam is bent into a circle, but the above theory applies to the most general type of bending, provided ρ is taken to be the radius of curvature of the neutral axis at the cross-section, the stress across which we are considering.

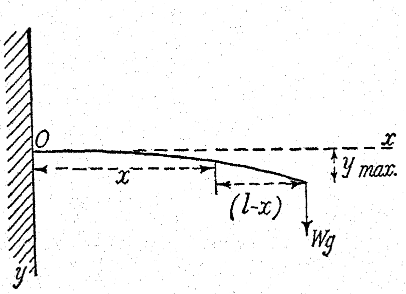


Fig. 11

We shall consider the simple **cantilever** shown in fig. 11. For simplicity we neglect the weight of the beam itself and suppose it to be clamped at one end, and loaded with a weight W at the other. We choose two axes, the x-axis lying horizontally in the position of the undeflected beam and the y-axis vertically downwards. The origin of co-ordinates is taken at O. Now by a well-known theorem in differential calculus, if the curvature is small,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.23)$$

Hence, equation (8.22) becomes

$$G = qAk^2 \frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.24)$$

Equation (8.24) is the fundamental equation for beam problems; since it is of the second order, we solve it by integrating twice. In this instance the couple, or bending moment, at any point x is $G = Wg(l - x)$. Hence

$$Wg(l - x) = qAk^2 \frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad (8.25)$$

Integrating twice, we have

$$A' + Wglx - \frac{Wgx^2}{2} = qAk^2 \frac{dy}{dx}, \quad . \quad . \quad . \quad (8.26)$$

and

$$B + A'x + \frac{Wglx^2}{2} - \frac{Wgx^3}{6} = qAk^2y, \quad . \quad . \quad (8.27)$$

where A′ and B are constants of integration, the values of which depend on the problem in hand. With our simple cantilever, since there is no depression at the origin, $y = 0$ when $x = 0$. Substituting these values in (8.27), $B = 0$. Again, the cantilever is horizontal at the origin so that when $x = 0$, $dy/dx = 0$; hence from (8.26), $A' = 0$. For this case, therefore, the depression y at any point x along the beam is

$$y = \frac{Wgx^2(3l - x)}{6qAk^2}.$$

In particular at the end of the beam where $x = l$, the depression is given by

$$y_{\max} = \frac{Wgl^3}{3qAk^2}. \quad . \quad (8.28)$$

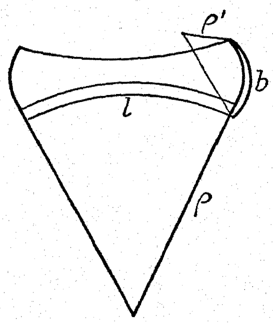


Fig. 12

For a beam of rectangular cross-section $Ak^2 = bd^3/12$, where b is the breadth and d the depth; for a circular cross-section it is $\frac{1}{4}\pi r^4$, where r is the radius. Since all the quantities in (8.28) except q may be measured, this affords a **method of measuring Young's modulus.**

If a bar of rectangular cross-section is used and bent by a symmetrical couple as shown in fig. 12, besides the curvature ρ in the plane of the paper there is an anticlastic curvature of radius ρ' developed in a plane perpendicular to this. It has been shown that the hori-

zontal strain e at any distance z from the neutral axis is z/ρ, where ρ is the radius of curvature of the axis. The *lateral contraction* f is similarly given by z/ρ'. Hence **Poisson's ratio** $\sigma = f/e = \rho/\rho'$. The radii may be determined directly by clamping pointers to the rod and observing the distances and angles traversed when a given couple is applied.

9. Energy in a Strained Body.

(*a*) A Stretched Wire.

The energy in strained systems is obtained by calculating the work done in straining the system. We suppose the strain to be produced in successive small steps, and apply to each step the general formula:

$$\text{Work done} = \text{force} \times \text{distance} = \text{stress} \times \text{area} \times \text{distance}, \quad (8.29)$$

so that the calculation in each case involves an integration. Applying (8.29) to a vertical wire, if the strain at any instant is e, the extension is Le where L is the unstretched length. The stress is qe. Hence the work done in stretching to the final strain e_0 is

$$V = \int_0^{e_0} qe \,.\, A \,.\, Lde = \tfrac{1}{2}qLAe_0^2 = \tfrac{1}{2}\frac{qAl^2}{L}, \quad . \quad . \quad . \quad (8.30)$$

where A is the area of cross-section of the wire, and l is the final extension, given by $l = Le_0$. If the stretch is produced by a weight W, since $q = WgL/(lA)$, therefore

$$V = \tfrac{1}{2}Wgl = \tfrac{1}{2}\frac{W^2g^2L}{qA}. \quad . \quad . \quad . \quad . \quad (8.31)$$

Now the loss in potential energy which the weight experiences is Wgl, so that from (8.31) we see that the energy stored up in the wire is only half that lost by the weight. The reason for this is that the other half has been converted into vibrational energy as we shall see later in discussing the oscillations of a spiral spring in § 11(*c*). With the wire, these vibrations are very rapidly damped by internal friction, and the energy is converted into heat, a slight rise taking place in the temperature of the loaded wire.

*(*b*) Rod or Wire under Torsion.

If the couple applied to the wire in § 5 is Q, the work done in twisting the wire through a further angle $d\phi$ is

$$dV = Qd\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.32)$$

Now from equation (8.9) $Q = \pi n\phi R^4/(2l)$, so (8.32) becomes

$$dV = \tfrac{1}{2}\frac{n\pi R^4}{l}\phi\, d\phi,$$

and the energy stored in the wire is

$$V = \int dV = \frac{n\pi R^4}{2l}\int_0^{\phi_0}\phi\, d\phi = \tfrac{1}{4}\frac{n\pi R^4 \phi_0{}^2}{l} = \tfrac{1}{2}Q_0\phi_0 = \frac{Q_0{}^2 l}{n\pi R^4}, \tag{8.33}$$

where ϕ_0 is the final angle of twist, and Q_0 the final steady couple. If the steady couple Q_0 is applied from the start, the total external work done is $Q_0\phi_0$, which is twice the energy stored in the wire. The other half of the energy is exhibited as *angular* or *torsional oscillations* which are eventually damped by internal friction.

***(c) Bent Beam.**

Considering a filament of the beam of area α at a distance z from the neutral axis, and applying the same reasoning as in sections (*a*) and (*b*) we find for dV, the energy in the filament,

$$dV = \int_0^{e_0} q\alpha \,.\, Le \,.\, de = \tfrac{1}{2}qL \,.\, \alpha e_0{}^2, \quad . \quad . \quad . \quad (8.34)$$

where e_0 is the final strain in the filament, and the unstretched length of the filament is L. Integrating (8.34) over the cross-section, and remembering that $e_0 = z/\rho$, we find

$$V = \int dV = \frac{qL}{2}\Sigma\alpha e_0{}^2 = \frac{qL}{2\rho^2}\Sigma\alpha z^2; \quad . \quad . \quad . \quad (8.35)$$

whence, since $\Sigma\alpha z^2 = Ak^2$, and from (8.22) the applied couple $G = qAk^2/\rho$,

$$V = \frac{qL}{2\rho^2}Ak^2 = \tfrac{1}{2}\frac{GL}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad (8.36)$$

The angle ϕ subtended by the beam at the centre of curvature equals L/ρ, hence (8.36) may be written

$$V = \tfrac{1}{2}G\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.37)$$

which shows its similarity with the torsional equation (8.33). In a similar manner the total work done on bending is $G\phi$, the remaining work $\tfrac{1}{2}G\phi$ being converted into oscillations which are eventually damped.

***10. Spiral Springs.**

We require an expression for the depression x of a flat spiral spring when it is loaded with a weight W. If the radius of the cylinder on which the spring is wound is a, and the radius of the wire itself is R (see fig. 13), we may regard the spring, to a first approximation, as a long wire of length l, fixed at one end and subjected to a torsional couple Wga at the other. This torsional couple results in the end being twisted through an angle ϕ, and, from fig. 13, the depression $x = a\phi$. Now by equation (8.9)

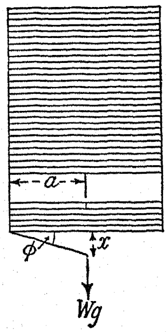


Fig. 13

$$Q = Wga = \tfrac{1}{2}\,\frac{n\pi R^4\phi}{l},$$

and therefore

$$x = \frac{2Wga^2l}{\pi n R^4}. \qquad (8.38)$$

The student should realize that this treatment is only approximate, bending as well as torsion taking place in the wire of which the spring is composed.

***11. Vibrations of Stretched Bodies.**

We shall consider the vibrations set up when a loaded cantilever, a twisted wire and a loaded spiral spring are given small displacements.

(*a*) Light Cantilever.

Let the cantilever on p. 84 suffer a displacement y at the end, and then be released. If y_1 is the displacement at time t after the release, then the force F which, if applied at the end, would keep the beam in equilibrium with this displacement is, from (8.28), given by the equation

$$y_1 = \frac{Fl^3}{3qAk^2}.$$

The *restoring force* acting on W is equal to F, but reversed.

Since the mass on the end is W, it experiences an acceleration given by

$$F = -W\,\frac{d^2y_1}{dt^2}. \qquad (8.39)$$

Hence (8.38) becomes

$$\frac{d^2y_1}{dt^2} = -\,\frac{3qAk^2}{Wl^3}\,y_1, \qquad (8.40)$$

and this equation is of the well-known type of (3.20) where $p^2 = 3qAk^2/Wl^3$. The time of oscillation is therefore

$$t = 2\pi/p = 2\pi\sqrt{\frac{Wl^3}{3qAk^2}}. \quad . \quad . \quad . \quad . \quad (8.41)$$

Equation (8.41) allows Young's modulus q to be determined for the beam if the remaining quantities in the equation are measured. It therefore affords a *dynamical method for Young's modulus.*

(*b*) Torsion Wire.

Consider a regular body fixed rigidly to the bottom of a torsion wire, the latter being fixed at the upper end. Then if the body is twisted through an angle ϕ in a horizontal plane, the *restoring couple* is by (8.9)

$$Q = \frac{\pi n R^4}{2l}\phi,$$

and hence from (4.17) the body will experience an angular acceleration given by

$$\frac{\pi n R^4}{2l}\phi = -I\frac{d^2\phi}{dt^2}, \quad . \quad . \quad . \quad . \quad . \quad (8.42)$$

where I is the moment of inertia of the body about the vertical axis of the wire. Since equation (8.42) is of the same type as (8.40), the period of angular vibration is given by

$$t = 2\pi\sqrt{\frac{2lI}{\pi n R^4}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.43)$$

an equation which allows the rigidity modulus n to be determined by a *dynamical method.*

(*c*) Loaded Spring.

The total work done in stretching the vertical spring in § 10 is Wgx, and as in the previous example of a stretched wire, the energy stored in the spring may be shown to be one-half of this. Hence the energy stored is $V = \frac{1}{2}Wgx$, or, from (8.38), by eliminating Wg,

$$V = \frac{\pi n R^4 x^2}{4la^2}. \quad . \quad . \quad . \quad . \quad . \quad (8.44)$$

When the spring is vibrating, the energy stored in the spring is continually being converted into kinetic energy of the vibrating mass and vice versa. We propose to write down the equation for the *total energy* of the vibrating system, a quantity which by the conservation

of energy must remain constant throughout the motion. If the vertical velocity of the end of the spring is dx/dt when its displacement is x, the kinetic energy of the vibrating mass is

$$E_W = \tfrac{1}{2}W\left(\frac{dx}{dt}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (8.45)$$

The spring itself possesses finite mass w, and it remains to calculate an expression for its kinetic energy. Since the upper end of the spring is at rest we assume that the velocity of the spring increases uniformly from a value zero at the upper end to dx/dt at the lower end. If the total length of the spring is l, the velocity at a distance s from the fixed end is $(s/l)(dx/dt)$, so that the kinetic energy of a small element of the spring at the point s is $\frac{1}{2}m\,ds(s/l)(dx/dt)^2$ where m is the mass per unit length of the spring. The total kinetic energy of the spring is therefore

$$E_s = \int_0^l \tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 \frac{s^2}{l^2}\,ds = \tfrac{1}{6}w\left(\frac{dx}{dt}\right)^2, \quad . \quad . \quad . \quad (8.46)$$

where $w = ml$, the whole mass of the spring. Adding equations (8.44), (8.45) and (8.46), we find the total energy of the system, and since this is constant,

$$\tfrac{1}{2}(W + w/3)\left(\frac{dx}{dt}\right)^2 + \frac{\pi n R^4}{4la^2}x^2 = \text{constant}. \quad . \quad (8.47)$$

If we differentiate with respect to t, and divide by dx/dt, we obtain the equation of motion:

$$(W + w/3)\frac{d^2x}{dt^2} + \frac{\pi n R^4}{2la^2}x = 0. \quad . \quad . \quad . \quad (8.48)$$

This is of the form $\dfrac{d^2x}{dt^2} = -p^2x$, and hence the time of vibration is

$$t = \frac{2\pi}{p} = 2\pi\sqrt{\frac{W + w/3}{\pi n R^4/(2la^2)}}, \quad . \quad . \quad . \quad . \quad (8.49)$$

and we note that the *effective mass* of the spring is one-third of its total mass. Equation (8.49) gives yet another dynamical method for the *rigidity modulus*.

Finally, from equations (8.49), (8.44) and the one which precedes it,

$$t = 2\pi\sqrt{\frac{x}{g}} \qquad (8.50)$$

if we neglect the weight of the spring. Hence, measurement of the extension x of the spring together with the time of oscillation of the suspended mass allows an estimate of g, the acceleration due to gravity, to be made.

EXERCISES

1. Define *stress*, *strain* and *modulus of elasticity*. Give an account of the stress-strain relations for a number of substances subjected to longitudinal extension or compression.

2. Describe the experimental determination of Young's modulus of elasticity.

Find the energy stored in a stretched wire of area of cross-section 2 mm.^{2} and initial length 50 cm., if it is loaded with a mass of 1000 gm. and Young's modulus for the material of the wire is 10^{12} dynes/cm.^{2} [1203 ergs.]

3. How may the rigidity modulus be measured by (*a*) a statical method, (*b*) a dynamical method?

Obtain an expression for the period of torsional oscillation of a horizontal bar attached at its mid-point to a vertical torsion wire.

4. Deduce an expression for Poisson's ratio in terms of Young's modulus and the rigidity modulus for the material. Show also that for an isotropic material, Poisson's ratio must lie between $+\frac{1}{2}$ and -1.

5. How is the bulk modulus of a solid determined? Calculate the bulk modulus for a specimen of steel, given that Young's modulus and the rigidity modulus for the specimen are 21×10^{11} and 8×10^{11} dynes/cm.^{2}, respectively. [$18{\cdot}7 \times 10^{11}$ dynes/cm.^{2}.]

6. Obtain an expression for the bending moment acting on a uniform light rod when it is bent into an arc of a circle of radius ρ, in terms of its geometrical moment of inertia and Young's modulus for the material of the rod.

How may experiments based on this relation be used to determine (*a*) Young's modulus, (*b*) Poisson's ratio?

7. A light rod of length $2l$ rests symmetrically on two knife-edges a distance $2a$ apart. If a load W is placed at the centre, show that the ends of the rod rise a distance $\dfrac{Wg}{4qAk^2} \cdot a^2(l - a)$.

8. Show that if a vertical flat spiral spring is undergoing vertical oscillations with a mass M fixed to the end, the effective mass is increased above that for a spring of negligible mass by $m/3$ where m is the mass of the spring itself.

9. Find the maximum velocity with which a 500 gm. weight moves if it is attached to a light spiral spring and then released, given that the latter shows a steady extension of 1 cm. when loaded with 100 gm. [70 cm./sec.]

10. Find the period of oscillation of the 500 gm. mass in Q. 9 and the maximum extension of the spring. [0·45 sec.; 10 cm.]

CHAPTER IX

Hydrostatics

The essential property that distinguishes a fluid (that is, a liquid or a gas) from a solid is that the fluid cannot remain in equilibrium under a shearing stress. The force across any surface in the fluid, or across its boundary, must therefore be a normal pressure.

We shall begin with some of the properties of liquids at rest, leaving the properties of liquids in motion until a later chapter.

1. Fluid Pressure.

If we consider liquid contained in a hollow rectangular vessel as shown in fig. 1, the bottom of the container must be exerting an upthrust sufficient to balance the weight of the liquid down, since the forces exerted by the sides are everywhere horizontal. The same volume of different liquids will possess different weights, the ratio of the mass to volume being defined as the **density** of the liquid. Hence

$$\rho = \frac{M}{V}, \qquad \ldots \ldots \ldots \quad (9.1)$$

where ρ is the density of the liquid, M its mass and V its volume. The force on the bottom of the container may therefore be alternatively expressed as

$$F = Mg, \qquad F = \rho V g. \qquad \ldots \ldots \quad (9.2)$$

Since pressure is defined as force per unit area, if the area of the base is A, the pressure on it is

$$p = \frac{F}{A} = \frac{Mg}{A} = \frac{\rho V g}{A},$$

and, since $V = Ah$, where h is the height of the liquid in the container,

$$p = \rho g h. \qquad \ldots \ldots \ldots \quad (9.3)$$

The same proof may be applied to any vertical cylinder of fluid within the vessel. It follows that the vertical pressure at any depth d in a liquid is $\rho g d$, assuming that the density ρ remains constant at all depths. This is true for small depths, since liquids are not very compressible,

but at great depths the increase in density with depth must also be taken into account.

Consider now the equilibrium of a small triangular prism of the liquid shown in fig. 2. Let the average pressure on the three faces of

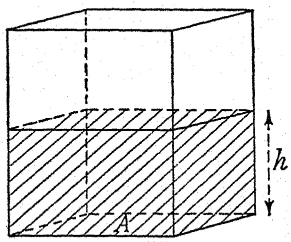


Fig. 1

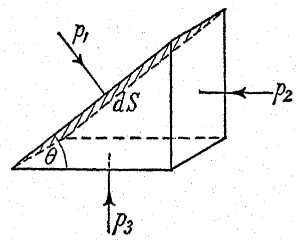


Fig. 2

the prism be p_1, p_2 and p_3 as shown. Then for equilibrium, if we equate forces horizontally and vertically,

$$(p_1 dS) \,.\, \sin\theta = p_2 \,.\, dS \sin\theta,$$

and
$$(p_1 dS) \,.\, \cos\theta = p_3 \,.\, dS \cos\theta.$$

Hence
$$p_1 = p_2 = p_3,$$

or the *liquid pressure is the same in all directions* and is therefore given by $\rho g h$ at any depth h in a liquid of density ρ.

The formula $p = \rho g h$ shows that the liquid pressure is the same at all points at the same level. This is easily proved independently by

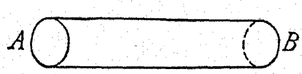


Fig. 3

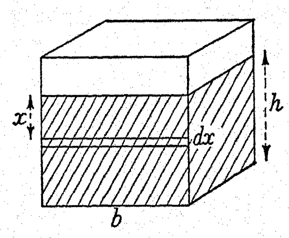


Fig. 4

considering a cylinder of fluid of small cross-section with its axis horizontal (fig. 3); for the only forces having a component in the direction of the axis are the forces on the ends A and B.

To calculate the total liquid pressure on one of the vertical sides of the vessels in fig. 4, consider a small strip of the side of height dx and breadth b at a depth x. The force on this, which will be normal to the strip, will be

$$dF = \rho g x \,.\, b\, dx$$

and the total force on the whole side will therefore be

$$F = \int dF = \rho g b \int_0^h x\,dx = \frac{\rho g b h^2}{2}. \quad . \quad . \quad . \quad (9.4)$$

Now the area of the vertical side under liquid is bh and the centre of gravity of this area is a distance $h/2$ below the surface. Considering (9.4), we see that

$$F = \frac{\rho g h}{2} \, . \, bh = \text{pressure at C.G.} \times \text{area of surface.} \quad (9.5)$$

Equation (9.5) represents a special case of a very useful general theorem. Suppose now the vertical side of the vessel were free to turn about a hinge at the bottom. Then, owing to the liquid pressure acting horizontally upon the side, unless the latter is held in position it will experience a couple about the horizontal axis of the hinge and will collapse outwards. To calculate this couple, we first consider the elementary couple dG due to the pressure on the small area $b\,dx$ at a distance $(h - x)$ from the base, that is

$$dG = \rho g x (h - x) b\,dx.$$

The total couple is therefore

$$G = \int dG = \rho g b \int_0^h x(h - x)\,dx = \frac{\rho g b h^3}{6}. \quad . \quad . \quad (9.6)$$

Now from (9.4), the total horizontal force F on the side is $\rho g b h^2/2$. Hence the total force may be considered to act at a distance

$$d = \frac{G}{F} = \frac{h}{3},$$

from the base. The point where this force may be considered to act is termed the **centre of pressure.** We observe that the centre of pressure of the vertical side is below the centre of gravity of the immersed portion of the side. The student should therefore note that while to calculate pressures on surfaces the pressure at the C.G. is found and multiplied by the area immersed, to calculate couples introduced by the liquid pressure, the pressure at the C.P. is found and multiplied by the appropriate distance from that point to the axis in question.

2. Archimedes' Principle.

The definition of density given in equation (9.1) applies to solids as well as to liquids; that is, it is the ratio of the mass to the volume of a body. Since mass and volume are measured in (pounds, cubic feet) and (gm., c.c.) on the F.P.S. and C.G.S. systems respectively,

different numbers are obtained for the density of a body according to the system of units applied. To avoid this complication, a quantity termed the **specific gravity** is sometimes used. This is defined as the ratio

$$\text{S.G.} = \frac{\text{weight of given volume of substance}}{\text{weight of an equal volume of water}},$$

and since the weights of both substance and water will be measured consistently in either pounds or grammes, the specific gravity will be a dimensionless number, independent of the units of measurements used. On the C.G.S. system, the gramme is defined as the mass of 1 c.c. of water at its temperature of maximum density. For this system, therefore, the density and specific gravity are numerically equal, although the former has dimensions ML^{-3}, whereas the latter is dimensionless.

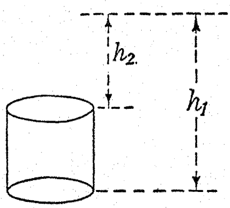


Fig. 5

Consider now the equilibrium of the immersed solid body of mass M and volume V shown in fig. 5. The immersion of the body will have resulted in a displacement of a volume of water V, equal to that of the body itself. If we take the simple case of a cylindrical body of uniform cross-sectional area A, its ends being horizontal, the vertical upthrust due to the difference in pressure on top and bottom will be

$$F = \rho g(h_1 - h_2)A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.7)$$

But $(h_1 - h_2)A$ is the volume of the body v and ρgv is the weight of the water displaced. Hence we see that **the immersed body experiences an upward thrust equal to the weight of water displaced** by the body. This is Archimedes' Principle. It holds for a body of any shape; to prove this, consider the body removed and the space filled with the liquid. The liquid would be in equilibrium under the surrounding pressure, which would be the same as before. The surrounding pressure as a whole must therefore be just sufficient to maintain the liquid in equilibrium, and must therefore be statically equivalent to the weight of the liquid, which passes through its C.G. This same vertical upthrust will continue to act if the actual body is now placed in position. The condition of *equilibrium* of an immersed body is therefore that the weight of the body down shall equal the thrust of the displaced liquid up, and be in the same line. Hence, if the body, of mass M, floats with a volume v immersed,

$$Mg = v\rho g, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.8)$$

and if σ is the density of the body and V its total volume, since $\sigma = M/V$,

$$V\sigma = v\rho,$$

or

$$\frac{v}{V} = \frac{\sigma}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.9)$$

Since ice has a density about 0·92 gm./c.c., and the density of sea-water is somewhat greater than unity, the fraction of an iceberg v/V which is immersed is about 9/10 the total volume of the iceberg.

3. Determination of Densities.

We shall consider the determination of the densities of gases and vapours in Part 2, confining our attention at present to the densities of liquids and solids.

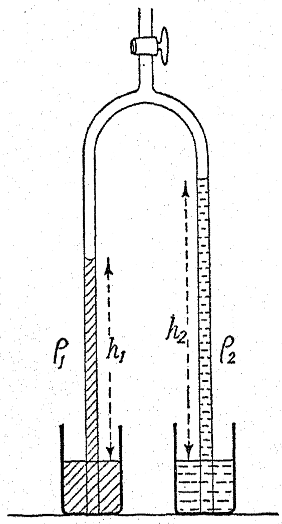


Fig. 6

(*a*) **Liquids.**

Of the many methods available, the determination of the relative masses of equal volumes of the liquid and of water respectively is the most direct. The vessel used is termed a **specific gravity bottle.** Another method involves the use of a vertical U-tube in the form known as **Hare's apparatus** shown in fig. 6. The liquid and water respectively are contained in two beakers into which dip the two arms of the inverted U-tube. Suction is then applied at the top, after which the top is closed. The two liquids will now be observed to stand at different heights, but since they are in equilibrium, their pressures down plus the common gas pressure on their meniscuses are equal in both cases to the external atmospheric pressure (see p. 105).

Hence

$$\rho_1 g h_1 = \rho_2 g h_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.10)$$

where ρ_1 and ρ_2 are the densities of the two liquids standing at heights h_1 and h_2 respectively.

The third method we shall describe involves the determination of the loss in weight of a body of known mass and volume suspended from a balance arm, when the body is surrounded by the liquid. The arrangement is shown in fig. 7. If the weights of the body in air and liquid are w_1 and w_2 respectively, the loss in weight $(w_1 - w_2)$ is, by Archimedes' principle, equal to the weight of liquid displaced, that is

$$(w_1 - w_2)g = \rho g v, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.11)$$

where ρ is the density of the liquid and v is the volume of the body. Such an arrangement is used to find the variation of the density of water with temperature, described in Part II. When the densities of a variety of liquids are known, an instrument known as the **common hydrometer** may be calibrated and used for determining densities. As shown in fig. 8 it consists of a uniform hollow glass tube at the end of which is a weighted bulb. The hydrometer is allowed to float in the liquid and according to the density of the liquid so the depth of the stem will change until the weight of the displaced liquid is sufficient to balance the constant weight of the hydrometer. The stem is then graduated with different densities corresponding to different depths of immersion.

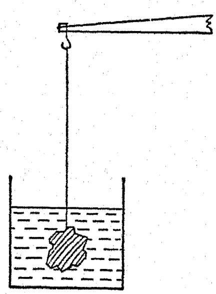


Fig. 7

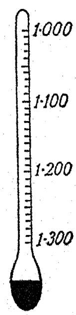


Fig. 8

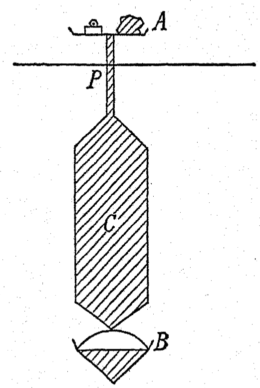


Fig. 9

To determine the density of a solid, variations of the specific gravity bottle method may be used. If the solid is insoluble in water, a certain mass of the solid is introduced into the specific gravity bottle, the remaining space being filled with water. The calculation is left to the student.

If the substance is soluble in water, its density may often be determined with reference to some liquid in which it is insoluble, say methylated spirit, after which the density of the latter may be compared with that of water.

The density of a solid which is insoluble in water may also be determined from (9.11), for since ρ is equal to unity and w_1 and w_2 may be measured, v can be calculated and the density of the solid $\sigma = w_1/v$. The experimental arrangement is sometimes made in a convenient form known as **Nicholson's hydrometer.** This consists, as shown in fig. 9, of a hollow cylindrical air-tight container C carrying upper and lower scale-pans A and B respectively. A fixed mark is made on the stem at some point P and the following measurements are made.

Let the weight which must be placed in the top scale-pan, which is in air, in order to sink the hydrometer so that the mark shall lie in the surface of the liquid, be w_1. Now replace w_1 by a given piece of the solid substance whose density is required, and note the weight w_2 which must be added again to sink the hydrometer to the mark. The weight of the body in air must then be $(w_1 - w_2)$. Finally, place the body in the water on the lower scale-pan and note the new weight w_3 which must be added to the top scale-pan again to sink the hydrometer to the mark. The weight of the body in water must then be given by $(w_1 - w_3)$. Hence the density of the body is

$$\sigma = \frac{(w_1 - w_2)}{(w_1 - w_2) - (w_1 - w_3)} = \frac{(w_1 - w_2)}{(w_3 - w_2)}, \quad . \quad (9.12)$$

for the denominator is the loss in weight of the body on being immersed in water, and this must equal the upthrust of the water which is the weight of water displaced, which in turn is equal to the volume of the body on the C.G.S. system. The density of a liquid can be found by Nicholson's hydrometer by observing the weights required to sink it to the mark in the liquid and in water respectively.

4. Stability of Floating Bodies.

Consider the floating body shown in fig. 10(a). It has been given a slight displacement and we observe that the two forces due to the weight of the body and the upthrust of the water respectively produce

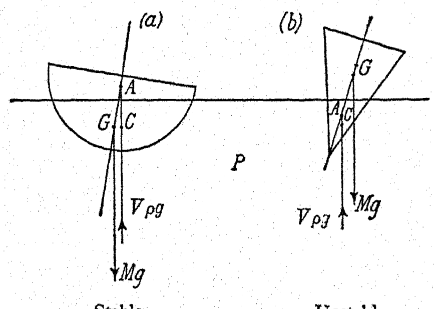


Fig. 10

a couple which tends to rotate the body back to its original position. The floating body is therefore in stable equilibrium. This holds so long as the centre of gravity of the body lies below the point A, where the vertical line from the centre of gravity C of the displaced liquid cuts the line of symmetry passing through the centre of gravity. C is called the **centre of buoyancy.** The point A is called the **metacentre**; to ensure

stable conditions the centre of gravity of the body must be as low as possible and the metacentre as high as possible.

With a completely submerged submarine, the volume of water displaced is the same for all orientations and the metacentre and the centre of buoyancy coincide. The condition for maximum stability is in this case simply that the centre of gravity shall be as low as possible.

The *metacentric height* H of ships, represented by AG in fig. 10, is found experimentally by observing the angle of tilt of the mast when weights are moved across the deck. If on moving a weight W a horizontal distance x, the ship heels through an angle α, then from fig. 10

$$Wxg = V\rho g \,.\, H \,.\, \sin\alpha,$$

where V is the volume of immersion. Since α is small, the equation may be written

$$H = \frac{Wx}{V\rho\alpha}.$$

It may be shown in the general case that

$$H = \frac{Ak^2}{V},$$

where A is the area of cross-section at the water-line, k is the radius of gyration of that area about the axis in it about which the body is displaced, and V is the original volume immersed before displacement.

5. The Hydraulic or Bramah Press.

This consists in principle of two vertical cylinders, one A of much larger area of cross-section than the other B, as shown in fig. 11. Its

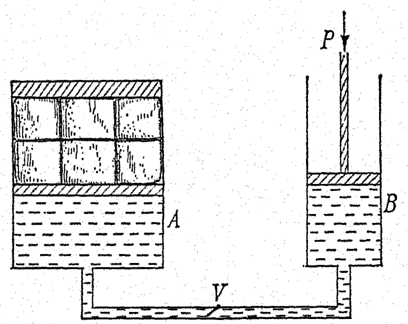


Fig. 11

purpose is to lift and sustain weights far greater than would otherwise be possible. The two cylinders are connected at the base by a horizontal tube containing a one-way valve V which allows liquid to flow

from B to A but not vice versa. On applying a moderate load P to a cylinder moving in B, the liquid between B and A is subjected to a pressure p given by $p = P/B$. The liquid transmits this pressure in all directions and, in particular, over the whole surface of A. The latter therefore experiences a force $W = pA = PA/B$, and is therefore able to sustain a load A/B times the load applied at B. The **mechanical advantage** of this machine is therefore $W/P = A/B$. If, however, it is required to lift the weight W a vertical distance h, then, by the principle of the conservation of energy, P must be moved a much greater distance d given by

$$Wh = Pd,$$

or

$$d = \frac{W}{P}h.$$

The ratio d/h is the **velocity ratio** (p. 38), and is clearly equal to the mechanical advantage.

6. Compressibility of Liquids.

We have so far assumed that liquids are incompressible, but at high pressures decrease in volume does occur. This was first shown by Canton in 1762, using an apparatus similar in principle to that shown in fig. 12, and termed a **piezometer.** The liquid is contained in the bulb A, and extends into the graduated capillary tube B, the upper end of which is connected to a compression pump and a manometer. The pressure is transmitted to the outside of A by liquid contained in the outer vessel D, which can be placed in communication with the compresser by the side-tube C and the tap E. This tap, together with the remaining taps F and G, allows the pressure to be communicated (1) to the outside only, (2) to the inside only, or (3) to the outside and inside simultaneously. While the last arrangement is all that is required to determine the compressibility, which is defined as the reciprocal of the bulk modulus K (see p. 77), it may be shown that if δv_o and δv_i represent the observed contractions under conditions (1) and (2) and δv_l under condition (3)

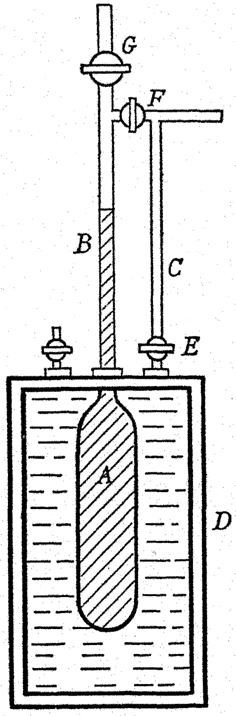


Fig. 12

$$\delta v_l = \delta v_o + \delta v_i,$$

if the containing vessel is isotropic. This last condition must be satis-

fied if a correction for the compressibility of the container is to be applied in the simple form

$$\frac{\delta v_l}{v_l} = P\left(\frac{1}{K} - \frac{1}{k}\right),$$

where k is the bulk modulus of the material of the container, v_l is the uncompressed volume of the liquid, and P is the applied pressure. Canton's apparatus, which was improved by Regnault, gave for many years the most reliable values of the compressibility of liquids, but the conditions have since been shown to be subject to serious error, and the values of K accepted at present are those due to Bridgman using an apparatus very similar to that described in Chap. VIII, § 6. The cylinder PQ of fig. 7 (p. 79) contains liquid instead of solid and is compressed by a piston rod which replaces the specimen AB. The displacement R_1R_2 of the sliding piston ring, which takes place when pressure is applied, is almost entirely due to the compression of the contained liquid, the compressibility of liquids being much greater than that of the solids of which the container and piston are made.

EXERCISES

1. What is meant by centre of pressure?

An equilateral triangle lies in a vertical plane with its vertex downwards and its base in the surface of a liquid. Find the position of the C.P. of the triangle. [$\frac{1}{2}h$.]

2. A sphere is floating with one-eighth of its surface above water; find its specific gravity. [245/256.]

3. A thin cylinder of area of cross-section A is closed at its lower end and contains water to a depth a. If it floats vertically in water when immersed to a depth b, find the depth of immersion if the water is baled out of the cylinder and the work done in baling. [$(b - a)$; $Aa(b - a)$.]

4. Write a short account of the methods available for finding the densities of solids and liquids. Give examples of the measurement of important physical quantities where a knowledge of densities is essential.

5. What determines the stability of floating bodies?

A hollow rectangular vessel is closed at both ends and loaded internally so that it floats in equilibrium with its longest sides vertical, when immersed in a liquid to a depth of 20 cm. Find the period of vertical oscillation if the vessel is depressed a small farther distance into the liquid and is then released. [0·9 sec.]

6. Describe the action of a hydraulic press.

If the platform of a hydraulic press has an area of 10 square feet, find the largest weight which can be sustained, given that the materials of a connecting tube of the press will burst under a pressure of more than 15,000 Kgm./cm.^{2} [$1{\cdot}37 \times 10^5$ tons.]

7. Write a short essay on the compressibility of liquids.

CHAPTER X

Properties of Gases and Vapours

1. Atmospheric Pressure.

The earth's surface is covered with a layer of air some forty miles high and the force per unit area which the weight of this air exerts is termed the *atmospheric pressure.* Unlike liquids, gases are highly compressible, and therefore the layer of air at the earth's surface, compressed by the weight of the superincumbent layers of air, is denser than the air above it. In fact, the density of the air decreases continually from the earth's surface upwards in an exponential manner. Owing to air currents, varying quantities of water vapour and other irregularities, the pressure of the atmosphere is continually changing, varying on either side of its mean value by about five per cent. Consider now the experiment shown in fig. 1 in which a number of glass tubes, closed at one end, have been filled with mercury and then inverted in an open trough also containing mercury. If the tubes are more than about 76 cm. long it is found that mercury will run out of the tubes until the vertical height of mercury is about 76 cm. This statement holds for the inclined tubes as well as for the vertical tube, the tops of the mercury columns all being observed to lie at one horizontal level. Now by the principles of the preceding chapter, the mercury columns must be exerting a pressure given by ρgh, where ρ is the density of mercury and h is the vertical height. This pressure is exerted down into the mercury in the trough, but this mercury will transmit the pressure equally in all directions. For equilibrium the pressure of the air downwards on to the general mercury surface, that is the atmospheric pressure, must therefore also be equal to ρgh. This experiment was first carried out by Torricelli, and the space above the mercury, which contains nothing except a little mercury vapour, is termed a **Torricellian vacuum.**

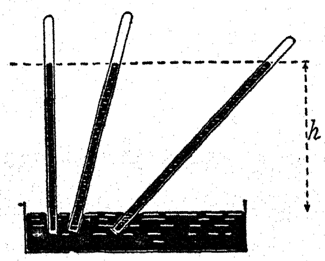


Fig. 1

In Chap. IX (at figs. 1 and 4, for example) nothing was said about

the effect of atmospheric pressure within the liquid. But it is easy to show, by the methods already used, that if a liquid is contained in an open vessel exposed to the atmosphere, and if p is the pressure at depth d, then

$$p = \Pi + \rho g d,$$

where Π is the atmospheric pressure.

It follows from this equation that the free surface is horizontal, for if $p = \Pi$, then $d = 0$. The equation may be regarded as stating that pressure applied at the surface of a liquid is communicated to every point within it.

2. Barometers.

The vertical tube in fig. 1 is termed a **simple barometer,** and the atmospheric pressure is usually expressed simply by h, the height of the mercury in the barometer tube. Owing to the variation of g, the acceleration due to gravity, the same barometric pressure may be balanced by different heights of the mercury column. Allowance has therefore to be made for this; the temperature must also be stated, for the density of the mercury decreases with rising temperature. It may be shown that if h_0 is the height which the mercury column would possess at 0° C. at sea-level in latitude 45°, then h_0 is connected with h, the observed height of the mercury column, by the equation

$$h_0 = h(1 - \alpha \cos 2\lambda - \beta H - \gamma t), \quad . \quad . \quad . \quad (10.1)$$

where $\alpha = 2{\cdot}65 \times 10^{-3}$, λ is the latitude, $\beta = 2 \times 10^{-7}$, H is the height in metres of the barometer above sea-level, γ is the coefficient of cubical expansion of mercury which is about $1{\cdot}82 \times 10^{-4}$ per °C., and t is the temperature in °C. An additional correction must also be applied for the **capillary depression** (see p. 119) of the mercury in the barometer tube. This varies with the width of the tube, and is a standard correction for barometer tubes of standard width. The standard barometer is known as **Fortin's barometer,** an illustration of which is shown in fig. 2. This consists of a glass barometer tube dipping into a reservoir the bottom of which is flexible and usually made of chamois leather. As the barometric height fluctuates, so the level of the mercury in the reservoir fluctuates, and since we are concerned with the height of the top of the mercury column above the surface level of the mercury in the reservoir it is convenient to adjust this level always to a constant value. An adjustable screw is therefore used to move a base-plate vertically in contact with the chamois leather until the level of mercury in the reservoir is always such that it just touches a vertical, inverted, fixed ivory pin-point. Coincidence of pin-point and mercury surface is judged by observing when the actual pin-point appears just to be in contact with its inverted mirror image in the mercury surface.

The height of the column is noted from a vernier scale moving over a fixed scale. Contact of the base of the moving scale with the top of the mercury column is judged by observing against the background of a white tile placed vertically behind the column. The scales are usually made of materials whose temperature coefficient of expansion is very small. If brass or steel scales are used, allowance must be made for their expansion in deducing the true height of the mercury column.

It will be realized that the determination of the absolute value of the atmospheric pressure may require tedious corrections. Further, on board ship, the motion of the ship introduces further difficulties owing to the "bumping" of the mercury column. In practice, therefore, a secondary barometer such as the **aneroid barometer,** a diagram of which is shown in fig. 3, is usually used. This, of course, requires calibration against a standard Fortin barometer, at some base station. The aneroid barometer consists of a flexible metal vacuum box, shaped in the form of a short cylinder of large area of cross-section.

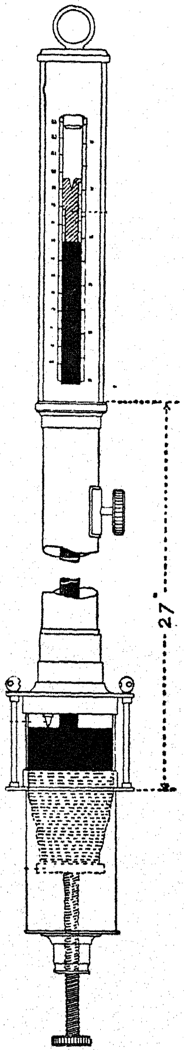


Fig. 2

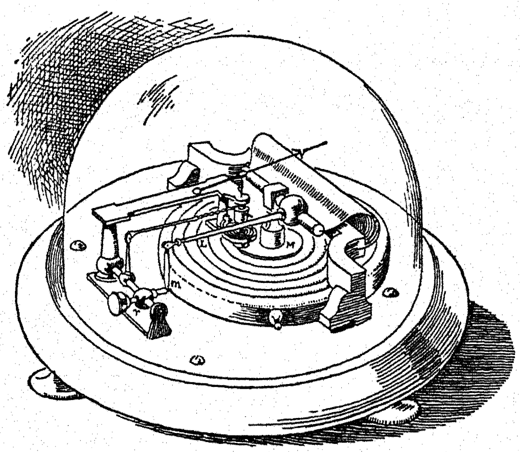


Fig. 3

This metal container is exhausted of air and then sealed. It is subject to atmospheric pressure outside, and is in a state of considerable strain. Any variation in the external pressure will therefore cause the cylinder to expand or contract, and owing to the elastic

properties of the metal of which it is composed, the movement of the cylinder accurately follows the change in atmospheric pressure. By a suitable system of levers the small changes in the movements of the cylinder are magnified sufficiently to move a pointer over a graduated scale so that the atmospheric pressure may be read off directly. The **barograph** is simply an aneroid barometer in which the movements of the cylinder are recorded by an attached pen on the surface of a rotating drum. A permanent record is thus obtained of the variations in atmospheric pressure throughout the day.

3. The Siphon.

The main reason for using mercury in barometer tubes is its high density, which allows the atmospheric pressure to be measured as a convenient height of about 76 cm. A tube containing water could be used, but its height would be about fourteen times that of the equivalent mercury column, while the pressure which its vapour would exert in the Torricellian vacuum would be inconveniently large. Consider now the behaviour of an inverted U-tube containing a liquid and having the shorter limb of the U dipping below the surface of a liquid contained in an open reservoir. Let the longer limb hang down as shown in fig. 4. It is found that liquid continues to run out of the longer

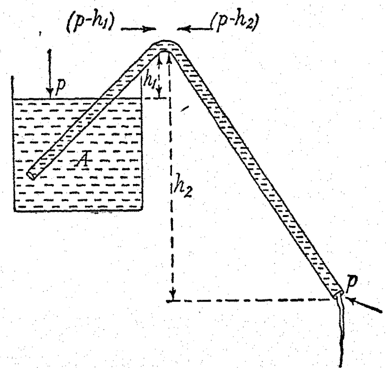


Fig. 4

limb until the level of the liquid in the reservoir falls below the end of the short limb. Such an arrangement is said to constitute a **siphon.** The pressure in the liquid just to the left of the bend is pressure transmitted by the liquid in A up the short limb of the tube. This will be the atmospheric pressure p which acts on A less the pressure of the vertical height h_1 of the liquid contained in the short limb. (For brevity we suppose p to be expressed as a height of the liquid.) Similarly the pressure at a point in the liquid just to the right of the bend will be that transmitted by the liquid up the longer arm of the siphon. This is the atmospheric pressure acting on the open end of the longer limb less the pressure of the vertical height h_2 of liquid in the longer limb. Since $(p - h_1)$ is greater than $(p - h_2)$ there is a resultant pressure driving the liquid along the tube, and this motion will continue until all the liquid in the reservoir has been emptied from above the bottom of the short limb. If the vertical height of the shorter limb is greater than the barometric height for the liquid

used, that is, if $h_1 > p$, the siphon will cease to work since the driving pressure is, as we have seen, $(p - h_1)$.

4. Pumps for Liquids.

(*a*) Suction Pump.

The operation of a suction-pump is shown in fig. 5. The pump consists of a vertical cylinder in which slides a piston operated by the pump-handle. On raising the piston, a vacant space—actually a vacuum—would be left below the piston. The valve at B therefore

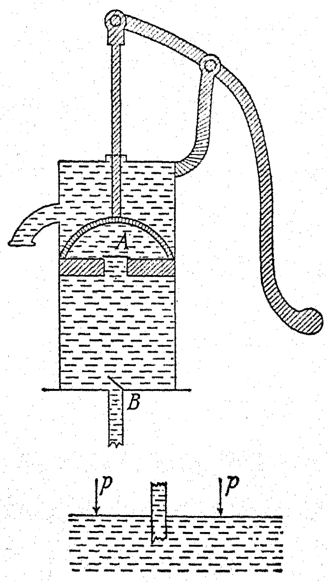


Fig. 5

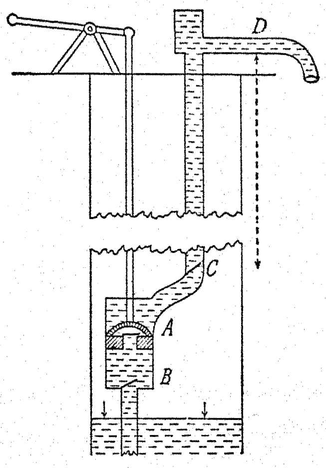


Fig. 6

opens upwards admitting liquid, since while there is no pressure above it, atmospheric pressure is transmitted through the liquid from the surface of the well and acts on the under surface of the valve. This process continues until the piston reaches the top of its motion. When the piston moves down, the valve B is pushed down and closed by the pressure transmitted through the liquid, but the valve A in the piston is opened by the pressure underneath it, and the liquid is therefore pushed out through the spout. We note that the space above the piston must be open to the atmosphere, otherwise a vacuum would be created here as the piston moved down.

(*b*) Force Pump.

If the surface of the liquid in the well is lower than the barometric height for the given liquid, then the pump will not work, since the

upward pressure is, as we have seen with the siphon, $(p - h)$, where p is the atmospheric pressure and h the height of the pipe above the surface of the liquid. There is therefore no point in making a water well deeper than about 34 feet if it is to be operated by a suction pump. On the other hand, by the use of a force pump the depth may be considerably increased. As shown in fig. 6 the force pump contains three valves A, B and C. The lower valve B is operated by atmospheric pressure just as for a suction pump, but the upper valve C is forced open by the upward movement of the pump rod controlled by the operator. This valve closes when the piston is moved down, owing to the weight of the head of water above it. The limit to the height which may be reached is determined by the strength of the operator, who has to overcome the downward pressure of the liquid column DC. If the pump is operated by a machine the limiting factor is the strength of the valves, and in practice the height through which water can be raised by a force pump is about 300 feet.

5. Boyle's Law.

Consider the apparatus shown in fig. 7. A quantity of dry air is enclosed in a glass tube and may be compressed by raising the mercury reservoir to which the glass tube is connected by indiarubber pressure tubing. When the level of the mercury is the same in both arms the pressure of the air enclosed in the tube must just equal the atmospheric pressure, otherwise the mercury would move under the pressure difference. If now the mercury is raised in the open reservoir it is found that the level of the mercury in the closed limb does not quite follow it, and in the position of equilibrium there is a steady difference in levels between the two columns of mercury. Calling this difference h, the pressure exerted by the enclosed air must be sufficient to sustain both atmospheric pressure p and also this height of mercury; it is therefore $(p + h)$. The volume of the enclosed gas may also be noted, if the tube has been calibrated; usually it is assumed that the tube is of uniform cylindrical bore, in which case the volume is proportional to the length of the air space. This is measured directly with a scale fixed to the apparatus: the scale also allows the height of the mercury in the open limb to be measured. As the pressure $(p + h)$ is increased, the volume v decreases, and pressure-volume measurements show that

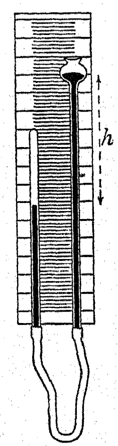


Fig. 7

$$(p + h)v = \text{constant.} \quad . \quad . \quad . \quad . \quad . \quad (10.2)$$

Equation (10.2) expresses *Boyle's Law*, which states that the *volume*

occupied by a gas is inversely proportional to its pressure, if the temperature is constant.

Since the mass m of the contained gas remains constant and the density ρ is defined as m/v, the density must vary directly as the pressure. In conducting the experiment, a moment or two must be allowed to elapse between adjusting the pressure to a new value and measuring the new volume. This is because alteration of the pressure on a gas causes its temperature to change, and Boyle's law is true only for constant temperatures. The temperature soon returns to its original value, and in this way a series of values of $(p + h)$ and v are obtained. It is usual to plot a graph of $(p + h)$ against $1/v$ whereupon a straight line is obtained passing through the origin. If the atmospheric pressure is not known, then, since it will remain constant over the time of the experiment, it is sufficient to plot h against $1/v$. A straight line is then obtained which cuts the axis of $1/v$ at a point where $h = -p$, thus allowing the atmospheric pressure to be determined indirectly. If $(p + h)$ is plotted against v instead of against $1/v$, the curve obtained is a hyperbola, since it corresponds to the curve $xy =$ const. We shall describe the determination of the density of gases and vapours in Part II, where also an account will be given of the deviations of gases from Boyle's law.

6. Air Pumps.

(*a*) Mechanical Pumps.

The simplest type of mechanical pump works on identically the same principle as the suction pump for liquids, described on p. 106, a vessel filled with air, which it is desired to evacuate, taking the place of the well. On raising the piston, a vacuum is created below it, the air below the valve exerts sufficient pressure to raise the valve and escapes into the cylinder space. This process continues until the piston reaches the top of its motion. When it descends, the gas beneath it is compressed and therefore, by Boyle's law, its pressure rises. The valve B therefore closes and as the pressure goes on rising it ultimately exceeds atmospheric pressure whereupon the valve A opens and the gas is expelled. To ensure that the joint between the piston and the cylinder shall be air-tight, liberal use is made of sealing oil. The limit to the vacuum obtainable is reached when the compression of the gas in the cylinder, which occurs at downstroke, is insufficient to raise the pressure of the enclosed gas above atmospheric pressure.

Gas pressure is usually expressed in mm. of mercury; a simple mechanical pump will not reduce the pressure much below 1 mm. of mercury. Modern mechanical pumps are very much more efficient; in fig. 8 are shown four stages in the process of evacuation by a Cenco-Hyvac pump. A cylindrical metal rotor A is mounted excentrically

in a cylinder. Through the wall of the outer cylinder slides a vane C; it is pressed in contact with A by a spring arm D. The vessel to be exhausted is connected to E and the exhaust is through the outward-opening valve at L. The pump is immersed in oil. In the first position (*a*), gas has just been admitted via E to the crescent-shaped space. In the second position (*b*), the gas has been compressed as the excentric rotor revolves, and fresh gas is admitted behind the rotor. Further compression follows in the stage (*c*), and finally at the stage (*d*) the

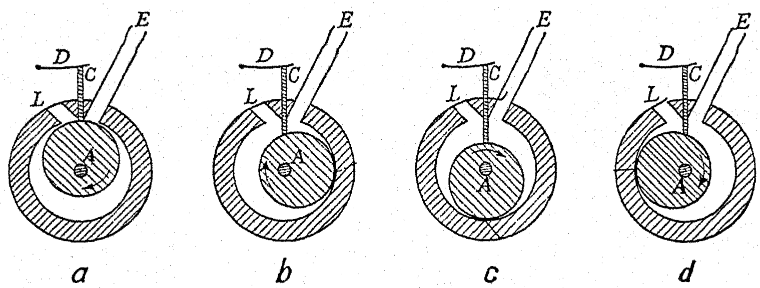


Fig. 8

valve L opens and the gas is expelled, for it has been compressed until its pressure is greater than atmospheric pressure. The pump is generally constructed in two parts, the first part exhausting directly to atmosphere and the second part supplying the gas to the first part. *The speed of pumping* is about 6 litres a minute and the vacuum obtainable about 10^{-3} mm. of mercury.

(*b*) **Liquid Pumps.**

In figs. 9, 10, 11 are shown diagrams of three pumps all of which depend on the motion of liquid past an orifice for their pumping action. In the **filter pump** the jet of water which issues from the tube A and passes down B entraps gas which has passed over from the vessel to be exhausted through the tube C. The process of trapping is continuous but the vacuum attainable is only a few cm. of mercury. This is partly due to the high vapour pressure of water.

A jet of mercury is much more efficient, and on this principle are based the Sprengel and Töpler pumps shown in figs. 10 and 11 respectively. In the **Sprengel pump** the arm A is raised so that mercury flows along the inverted U-tube. As the mercury passes the tube at the top of the U, which communicates with the vessel to be exhausted, it breaks up into pellets, and between the pellets is enclosed a small air-bubble. The exhausting process is extremely slow and laborious, as the mercury has to be collected and replaced in the reservoir periodically, but a vacuum as high as 10^{-5} mm. of mercury may eventually

be obtained. The **Töpler pump** works on a similar principle, except that the gas is allowed to expand into a fairly large bulb B, so that

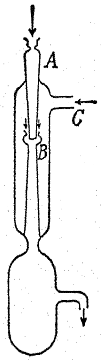


Fig. 9

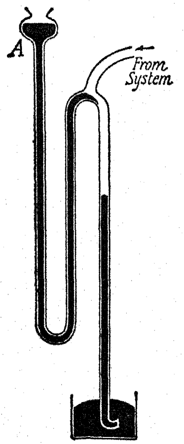


Fig. 10

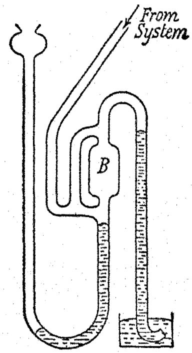


Fig. 11

the exhausting process is somewhat more rapid than with the Sprengel pump.

(*c*) Vapour Pumps.

For obtaining the highest possible vacua, such as are required in some wireless valves and X-ray tubes, instead of liquid as in the Sprengel and Töpler pumps, the passage of vapour past an orifice is used. In fig. 12 is shown a simple form of **condensation pump.** Mercury is boiled in the vessel A and mercury vapour passes out through the jet B. This orifice is surrounded by a Liebig's condenser C so that the mercury vapour jet is condensed, but not before gas expanding over from V, the vessel to be evacuated, has been caught in the unidirectional jet of vapour. The mercury is returned to the boiler A, but the gas escapes through the exhaust E. Such a pump will not exhaust directly from atmospheric pressure; it is therefore used in conjunction with, say, a Cenco-Hyvac pump. The vapour pump therefore is supplied with an initial "backing" vacuum supplied by its "backing" pump. It is then able to carry the vacuum to 10^{-6} or 10^{-7} mm. of mercury.

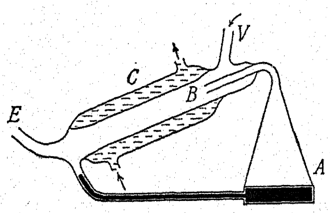


Fig. 12

(*d*) **Other Processes.**

To obtain vacua much higher than 10^{-6} mm. of mercury, **chemical processes** are usually used. Thus many gases are rapidly absorbed by coco-nut charcoal when the latter is surrounded by liquid air (see Part II). Hydrogen may be eliminated by its affinity for palladium or platinum black. Traces of oxygen and other gases are sometimes removed by "flashing". In this process, which is largely used in wireless valve construction, a metal such as magnesium or calcium is placed on the filament and suddenly vaporized by passing a large electric current momentarily through the filament. Much of the vaporized metal condenses as a bright mirror on the glass walls of the valve, but at the same time the oxygen or other gas has combined with some of the metal to form a compound with negligible vapour pressure.

7. Measurement of Gas Pressures.

Large pressures of several thousand atmospheres may be measured with the free piston gauge described on p. 80, the pressure being transmitted by some liquid buffer between gas and gauge. *Secondary gauges* are often used in practice, for example, to measure steam pressure. The simplest of these is the **Bourdon spring gauge,** which consists of a plane spiral of metal tubing flattened at the closed end. When the pressure is transmitted down the tubing the spiral tends to straighten out, and a pointer may be made to register the pressure.

Moderate pressures are measured with **manometers.** The common manometer is a U-tube containing mercury. One side is left open to the atmosphere, and the other limb is connected to the vessel the pressure of whose contents is required. The difference in height of the mercury in the two limbs added to the atmospheric pressure gives the total pressure in the vessel. To obtain an appreciable difference in level when the pressure is not very different from that of the atmosphere, a liquid of small density, such as paraffin oil, is used in the manometer instead of mercury.

To measure low pressures, such as those produced by the vacuum pumps described in this chapter, a great many manometers are available, depending on the different physical properties of gases at low pressures. For example, gauges based on the viscous friction of gases and on their thermal conductivity are available. The discussion of these is beyond the scope of this book (but compare Part II). Fortunately, however, they are all secondary gauges. They are calibrated against the **McLeod gauge,** the operation of which the student will readily understand. In fig. 13 is shown Gaede's modification of the McLeod gauge. Gas from the system whose pressure is required enters the gauge through B and fills it down to the level of the mercury reservoir

G. The reservoir is then raised, cutting off the gas present in the bulb H and compressing it into the capillary extension which lies along the scale KK_1. The mercury rises faster in the left-hand arm, and may be made to stand at any arbitrary height in the tube A above that in the closed tube. If the required gas pressure is p, and the total volume of H and the capillary extension is V, then applying Boyle's law to the mass of gas enclosed in this bulb and eventually compressed in the capillary, we have

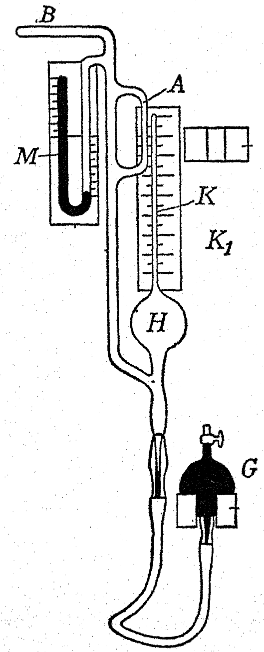


Fig. 13

$$pV = (p + h)V', \quad . \quad (10.3)$$

where h is the height of the mercury in the tube A above the level of the mercury in the capillary tube, and V' is the volume the gas now occupies in the capillary tube. Since h is a few cm. and p is much less than 1 mm., equation (10.3) may be written

$$pV = hV'$$

or

$$p = h\frac{V'}{V}. \quad . \quad . \quad (10.4)$$

Pressures from 100 mm. to 1 mm. can be read directly on the manometer M, which is simply an exhausted U-tube filled with mercury in the left-hand limb. Pressures from 1 mm. to 10^{-4} mm. may be read on the scale KK_1. We should note that the pressures registered by all gauges which contain liquids as the working substance register the vapour pressure of this liquid simultaneously with the pressure which it is desired to measure.

EXERCISES

1. Describe the construction of a standard mercury barometer, indicating what corrections are necessary to reduce the observed reading of atmospheric pressure to the accepted standard conditions.

2. Explain the action of (*a*) a force pump, (*b*) a suction pump.
Find an expression for the pressure inside a vessel of volume V (initially atmospheric pressure) at the end of the nth stroke of an ordinary mechanical air-pump, the volume of whose cylinder is v.

$$\left[P_n = p_1\left(\frac{V}{V+v}\right)^n\right].$$

3. How may a very low gas pressure such as that required in a vacuum electric filament lamp be obtained? By what means is such a low gas pressure measured?

4 State Boyle's law, giving two examples of practical applications of the law involving quantitative results.

5. A spherical diving chamber contains air initially at atmospheric pressure and has a small opening at the bottom of its lower surface. How deep is the centre of the chamber below a water surface when the water has risen so as to half-fill the chamber? [10·34 m.]

6. If the chamber in Q. 5 is 10 m. in diameter, what volume of air at atmospheric pressure must be pumped into the chamber so as to expel the water completely? $\left[\frac{500\pi}{3}\cdot\frac{15{\cdot}34}{10{\cdot}34}\text{ c. metres.}\right]$

CHAPTER XI

Properties of Liquids at Rest

1. Surface Tension.

If a small quantity of mercury is dropped on to a wooden bench, it is found that it splits up into a number of different-sized globules. The larger globules are oblate spheroids, while the smaller are nearly perfect spheres. This spherical shape is also shown by small water drops issuing from a jet. If the mercury drops are distorted from their equilibrium shape by slight pressure from the finger, it is found that they return to their original shape when the pressure is released. This behaviour is so like that of a child's rubber balloon that the phenomenon is ascribed to **surface tension**; that is, the liquid drop behaves as though there were skin under a tension stretched over its surface. This assumption is very useful, as it enables us to predict the behaviour of the surface forces in liquids in a great many instances, but we shall now proceed to show from the molecular hypothesis that the apparent existence of forces and tensions *parallel* to the surface really arises from forces in the liquid which act *normal* to the surface.

2. Molecular Theory of Surface Tension.

We assume that the molecules of a liquid exert an attraction on each other somewhat analogous to the gravitational attraction that exists between all bodies. The forces are actually electrical and not gravitational, as the student will understand by reference to Part V; gravitational forces are far too small to produce the observed experimental effects. If we knew the precise law of force according to which each molecule attracted the others, it would be possible to calculate mathematically the behaviour of liquids. Unfortunately the liquid state is too vague (being somewhere intermediate between the solid state, where the molecules have no translational motion, and the gas state, where the molecules have almost free translational motion) to allow us to formulate a definite law of attraction, but Laplace introduced a working hypothesis based on the conception of a **radius of molecular attraction.** According to this idea, each molecule is considered to exert attractive forces on the molecules surrounding it for a certain radius c, but beyond this distance the force of attraction is

considered to be zero. In actual fact the force of attraction must fall off continuously as the distance increases, but the admittedly approximate assumption of the radius of molecular attraction provides a very useful tool in qualitative and, in some cases, quantitative work. For example, consider a molecule A situated in the first instance at a distance greater than the radius of molecular attraction below the surface of the liquid as in fig. 1(a), and in the second instance at a distance less than the radius of molecular attraction from the surface as in fig. 1(b). Then in the first case the molecule will experience no resultant force as it is attracted equally in all directions, but in the second case there will be more molecules attracting it from below than

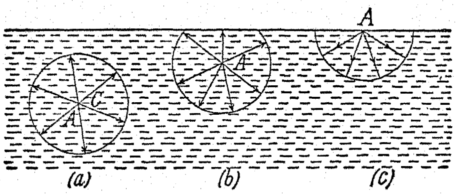


Fig. 1

from above. This downward force will be a maximum for molecules in the liquid surface since there is now a completely unbalanced hemisphere of forces acting from below on the molecule. The molecule will therefore experience a force normal to the surface of the liquid, and these normal forces acting over the whole surface of an isolated portion of the liquid such as constitutes a drop will cause that surface to take up a curved shape just as though there were an elastic skin producing a surface tension in the surface of the liquid (see end of § 3).

3. Definition of Surface Tension.

We shall use the convenient fiction of a tension acting in the surface of a liquid freely in our subsequent discussion. **Surface tension** is defined as the **force per unit length acting on either side of any line drawn in the surface of a liquid,** the direction of the force being tangential to the surface and perpendicular to the line. To illustrate this definition consider a soap-film supported by a wire frame as in fig. 2. If the frame is vertical and the wire AB is free to move, it will move down, stretching the film until the surface tension force up equals the weight of the wire down. Representing the mass of the latter by w, for equilibrium, we have

$$wg = 2Tl, \qquad \ldots \ldots \ldots \quad (11.1)$$

where l is the horizontal length of AB across which the surface tension T acts. The factor 2 is introduced since the film has two sides. From

(11.1) we see that surface tension is a *force per unit length*, and therefore has dimensions MT^{-2}.

Consider next the rectangular frame-work shown in fig. 3, and let this frame-work be horizontal. Let the cross-wire PQ, of length l, be pulled a distance x along the frame-work, thus increasing the area of each side of the film by $A = lx$. The work done, W, against the surface tension forces is

$$W = 2Tl \, . \, x = 2TA. \quad . \quad . \quad . \quad . \quad . \quad (11.2)$$

We see that (11.2) gives a second definition of surface tension, for since the total increase in area of the film is $2A$, the **surface tension** may be defined as the **work done in increasing the area of the film by unit amount.** This work done is stored up as energy in the stretched film and can be regained if the film is allowed to contract. *Surface*

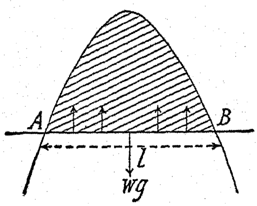


Fig. 2

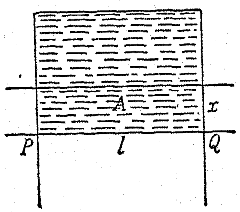


Fig. 3

tension and **surface energy** are thus closely connected. However, it is found experimentally that unless special precautions are taken, the value for T obtained by methods based on (11.2) is different from the value obtained by methods based on (11.1). The reason is that the second is a dynamical method, and when contractions and expansions of a film take place, temperature changes occur and **thermal energy** is also involved. The two methods give the same result if we stipulate that the energy change must be arranged to take place without change of temperature, that is, under **isothermal conditions.**

The concept of surface tension as energy per unit area affords us an explanation why small drops of liquid are spherical. All systems tend to the condition where their **total potential energy is a minimum,** and the body which has the least surface area (corresponding to the least surface energy) for a given volume is the sphere. Large drops are flattened into oblate spheroids because their gravitational energy would be greater if the centre of gravity of the drop were at the centre of a sphere of the same volume as the spheroid.

4. Pressure Difference over a Curved Surface.

Consider a small curvilinear rectangle ABCD (fig. 4, *a* and *b*) of the curved surface of a film. Let the sides have lengths δl_1 and δl_2

and let the radii of curvature of these sides be R_1 and R_2. Suppose the film is in equilibrium with an excess pressure p on one side of the film. To find the relation between the surface tension T and the excess pressure p, we apply the principle of virtual work; that is, we suppose the film to expand, and equate the work done by the pressure to the work done against the surface tension of the film as its area is increased. If the sides of the film increase by the fractions of their lengths, a_1 and a_2, the increase in the area of *one side* is

$$\delta A = \delta l_1(1 + a_1) \,.\, \delta l_2(1 + a_2) - \delta l_1 \,.\, \delta l_2,$$

so that the work done against surface tension is

$$\delta W = T \delta l_1 \delta l_2 (a_1 + a_2), \quad . \quad . \quad . \quad . \quad (11.3)$$

neglecting the small product $a_1 a_2$.

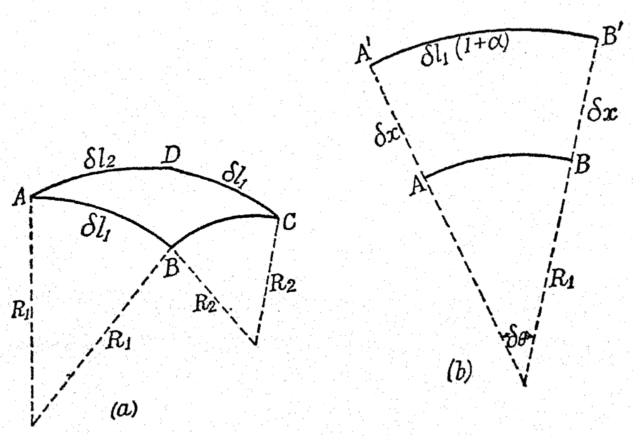


Fig. 4

The work done by the pressure is

$$\delta W = p \,.\, A \,.\, \delta x = p \,.\, \delta l_1 \delta l_2 \delta x. \quad . \quad . \quad . \quad (11.4)$$

But, from fig. 4(*b*),

$$a_1 \,.\, \delta l_1 = (R_1 + \delta x)\delta\theta - R_1 \,.\, \delta\theta = \delta x \,.\, \delta\theta = \delta x \,.\, \frac{\delta l_1}{R_1};$$

hence $a_1 = \delta x / R_1$, and similarly $a_2 = \delta x / R_2$.

Substituting these values of a_1 and a_2 in (11.3), and comparing with (11.4), we find

$$\boldsymbol{p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (11.5)$$

Equation (11.5) is very useful and allows us to calculate, for example, the excess pressure which must exist inside a spherical soap-bubble. In this case $R_1 = R_2 = R$, and there are two sides to the film, so (11.5) becomes

$$p = \frac{4T}{R}. \qquad \qquad (11.6)$$

The excess pressure therefore varies inversely as the radius of the bubble, and if two bubbles of different radii are connected by a common tube the larger bubble will grow at the expense of the smaller since the air pressure in the latter is greater than that in the former.

5. Angle of Contact.

If water is contained in a narrow vertical glass tube, it is found on examination that the surface of the water is not level, but is higher at the edges where it is in contact with the glass than at the centre. The attraction between the water molecules and the glass is greater than the attraction between water molecules themselves. With mercury the reverse is true and consequently the surface is convex. If careful examination is made of the behaviour of the liquid where it meets the solid it is found that with mercury the two surfaces meet at an angle of about 160°. This angle is said to be the **angle of contact.** For a liquid like water which wets the surface the angle of contact is zero. Measurement of the angle of contact may be carried out with Ablett's apparatus shown in fig. 5. A cylinder with its axis horizontal is lowered into the liquid until the liquid surface appears horizontal at the line of contact with the cylinder. Then

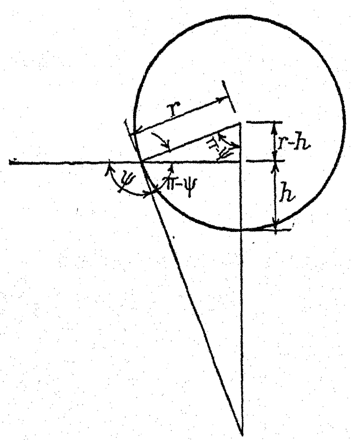


Fig. 5

??

$$\cos\psi = \frac{h - r}{r}, \qquad \qquad (11.7)$$

where ψ is the angle of contact. Analogous measurements are taken with the cylinder rotating in clockwise and anticlockwise directions so that a mean of two values may be obtained, the values corresponding to liquid advancing or receding over the surface of the solid.

6. Methods of Measuring Surface Tension.

(*a*) Capillary Tube Method.

If a fine capillary tube is placed vertically with one end beneath the surface of a liquid, either a rise or a depression of the liquid in the tube occurs with respect to the general level of the liquid outside the tube. The rise occurs with liquids whose angle of contact ψ is less than 90°; a diagram of such a case is shown in fig. 6. When the liquid column is in equilibrium the weight of the liquid column acting downwards must be just balanced by the *surface tension force* or *capillary attraction* upwards. The liquid column consists of the cylinder of volume $V = \pi r^2 h$, and the meniscus, which may be shown by elementary mensuration to have a volume $v = \pi r^3 (\sec\psi + \frac{2}{3}\tan^3\psi - \frac{2}{3}\sec^3\psi)$. This assumes that the surface of the liquid is in the shape of a spherical cap. Hence, equating the two forces, we have

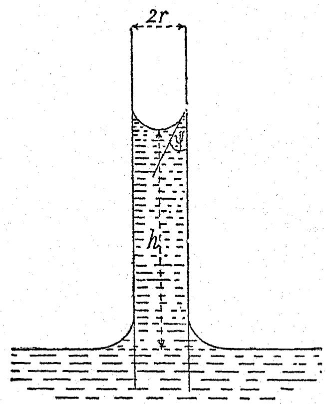


Fig. 6

$$2\pi r T\cos\psi = \pi r^2 h\rho g + \pi r^3 \rho g(\sec\psi + \tfrac{2}{3}\tan^3\psi - \tfrac{2}{3}\sec^3\psi), \quad (11.8)$$

where ρ is the density of the liquid. This gives

$$T = \frac{r\rho g\sec\psi}{2}\{h + r(\sec\psi + \tfrac{2}{3}\tan^3\psi - \tfrac{2}{3}\sec^3\psi)\}. \quad (11.9)$$

If the liquid wets the surface the meniscus is approximately a hemisphere, and $\psi = 0$, so (11.9) becomes

$$T = \frac{\rho r g}{2}\left(h + \frac{r}{3}\right). \quad . \quad . \quad . \quad . \quad . \quad (11.10)$$

In carrying out the experiment the tube must be thoroughly cleansed before use. The height h is determined with a vertically travelling microscope. The radius of the capillary tube is usually found by drawing a thread of mercury into the tube, measuring the length l of the mercury column and then running the thread of mercury out into a watch-glass and finding its weight w. Then, if σ is the density of mercury, $\pi r^2 l\sigma = w$. As the radius particularly required is that at the point where the meniscus is formed, it is

better to mark the tube at this point and then to break the tube here at the conclusion of the experiment. The radius is then found with a travelling microscope for several diameters mutually at right angles, the average value being taken. The method is fairly accurate for liquids which wet the tube; otherwise the angle of contact must be known. If very accurate values are required the exact shape of the meniscus must be examined. The curved surface of a liquid between its flat general level and the line of contact with the solid is sometimes referred to as the *capillary curve.*

The surface tension between two immiscible liquids is termed the *interfacial surface tension.* It may be determined by placing a layer of one liquid on top of the other and inserting a vertical capillary tube at the junction.

(*b*) Jäger's Method.

In this method bubbles are blown beneath the surface of the liquid by an apparatus as shown in fig. 7, the pressure at which the bubbles are formed being given by the manometer. If we assume that the

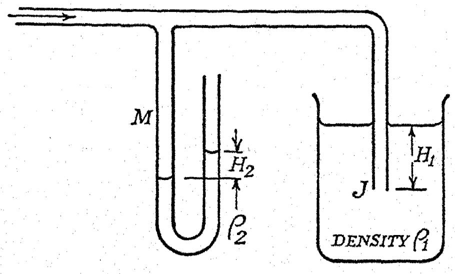


Fig. 7

bubbles become detached when they are just greater than hemispheres of radius r equal to that of the tube, then by equation (11.5) the pressure registered by the manometer is

$$p = \frac{2T}{r} = \rho_2 g H_2 - \sigma_1 g H_1, \quad . \quad . \quad . \quad (11.11)$$

where ρ_2 is the density of the liquid in the manometer, and H_2 is the difference between its levels. The quantities σ_1 and H_1 are the density of the liquid in which the bubbles are being blown, and the depth of the bubble orifice below the surface of the liquid. The method has the advantage over the capillary tube method that the liquid-air interface is continually being renewed so that contamination of the surface, a factor which makes a great difference to the surface tension, is avoided.

The method may also be used to determine the surface tension of molten metals.

(*c*) Frame Method.

A vertical rectangular metal frame is supported from a rigid horizontal rod and dips into the liquid as shown in fig. 8. The rod is fixed to a horizontal torsion wire which acts as an axis, suitable counterpoise being placed on the other side of the rod. The dish containing the liquid is then lowered and the frame follows it, twisting the torsion

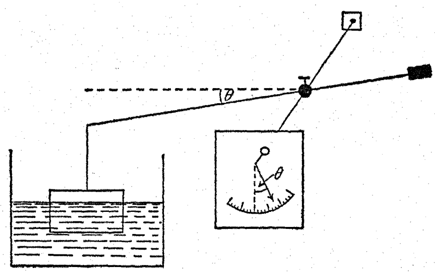


Fig. 8

wire through an observed angle θ. When the torsional couple is just greater than the pull down due to surface tension, the frame jerks out of the liquid. The maximum angle of twist is observed and then suitable weights are suspended in place of the balance arm to produce the same couple as that produced by the surface tension forces. If the equivalent effective mass is m,

$$2Tl = mg,$$

where l is the length of the horizontal wire of the frame. There are many variations of this type of apparatus, sometimes flat wire rings or horizontal disks being used instead of the wire frame.

(*d*) Drop Weight Method.

In this method, which is valued in industry for its simplicity and rapidity, liquid is allowed to drop from a circular orifice and the average weight of a drop is determined by counting a given number and finding the total weight of liquid collected. Then if we assume that the drop breaks away immediately after the issuing liquid becomes hemispherical, we have

$$T2\pi r = mg, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.12)$$

where m is the mass of the drop, and r is its radius, which is taken to

be that of the tube. Actually equation (11.12) does not exactly represent the true state of affairs, but for a given tube

$$mg = KT,$$

where K is a constant which must be determined for the tube by calibration with a liquid of known surface tension.

(e) Stationary Drops or Bubbles.

We consider either a stationary drop as shown in fig. 9(a), or a stationary bubble as shown in fig. 9(b), each being in contact with a horizontal surface. If the drop or bubble is large and flat, and we consider the equilibrium of the section shown in fig. 9(c), the total force per unit length to the right is $T(1 - \cos\psi)$. This balances the

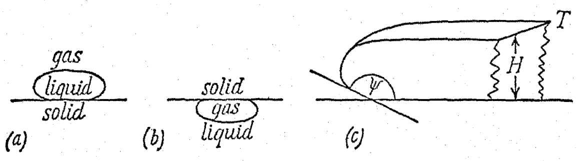


Fig. 9

hydrostatic pressure across the section of the drop, and if H is its mean thickness this gives rise to a force per unit length of $g\rho \dfrac{H}{2} \cdot H$. Hence

$$T(1 - \cos\psi) = \frac{g\rho H^2}{2}. \qquad . \quad . \quad . \quad . \quad (11.13)$$

This method involves a separate measurement of the angle of contact ψ; H is usually determined with a microscope.

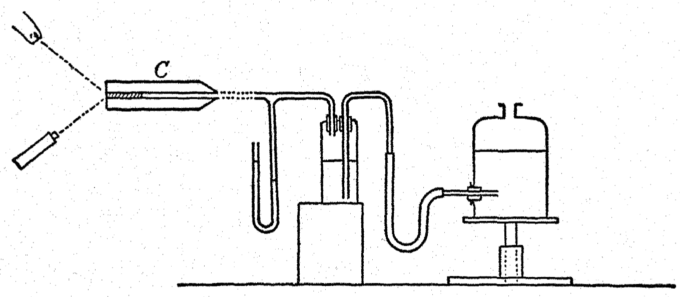


Fig. 10

(f) Ferguson's Method for a small quantity of Liquid.

The arrangement, which is a variation of the capillary tube method, is shown in fig. 10. The capillary tube C, which is very narrow, is

arranged horizontally, and a pressure p is applied by raising the reservoir on the right until the liquid, which is placed in the capillary, has its surface forced plane. This condition is judged by the occurrence of regular planar reflection of light incident on the end of the tube. If the angle of contact of the liquid is ψ and the radius of the tube at the meniscus is r, then

$$2\pi r T \cos\psi = p \,.\, \pi r^2 = \rho g h \,.\, \pi r^2, \quad . \quad . \quad (11.14)$$

where ρ is the density of the liquid in the attached manometer, and h is the difference in levels of the arms. The required quantities in (11.14) are measured in the manner already described for the vertical capillary tube method.

* (g) **Ripple Method.**

This is an elegant dynamical method due to Lord Rayleigh. In fig. 11 is shown a section of a wave across the surface of a liquid. We assume the waves to be simple harmonic in form (see Part IV); then the vertical displacement y at any point x is given by

$$y = a \sin\left(\frac{2\pi x}{\lambda} + b\right), \quad . \quad . \quad . \quad . \quad (11.15)$$

where a is the amplitude, λ is the wavelength, and b is a constant

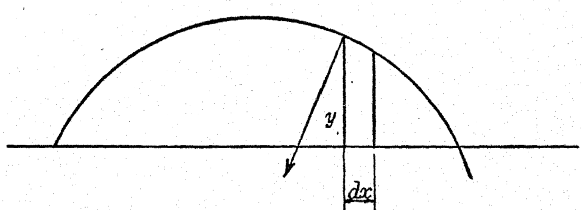


Fig. 11

termed the phase constant. If r is the radius of curvature of the portion of the wave at y, then using equations (8.23) and (11.15) we find

$$\frac{1}{r} = \frac{d^2y}{dx^2} = -\frac{4\pi^2 y}{\lambda^2}. \quad . \quad . \quad . \quad . \quad (11.16)$$

Now, if we consider the effect of surface tension, a pressure $p = T/r$ will by equation (11.5) act normally to the curve, the other radius being infinite since we treat the waves as cylindrical rollers. Hence, (11.16) becomes

$$p = -\frac{4\pi^2 y}{\lambda^2} T,$$

and the vertical *downward* force on an area of unit length perpendicular to the paper and of length dx in the x-direction is

$$p\,dx = +\frac{4\pi^2 y}{\lambda^2} T\,dx. \qquad (11.17)$$

Now the force arising from the hydrostatic pull downwards on this element of volume due to the action of gravity is $g\rho y\,dx$, where ρ is the density of the liquid. The total downward force on the element of volume is therefore

$$\rho y\,dx\left(g + \frac{4\pi^2 T}{\lambda^2 \rho}\right). \qquad (11.18)$$

Surface tension is therefore shown by equation (11.18) to increase the effective acceleration due to gravity by an apparent amount $4\pi^2 T/\lambda^2\rho$.

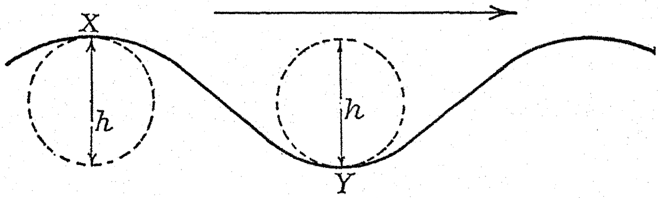


Fig. 12

Now the velocity of propagation c of purely gravitational waves is, as we shall show later, given by

$$c = \left(\frac{g\lambda}{2\pi}\right)^{\frac{1}{2}}, \qquad (11.19)$$

and hence, when surface tension is also operative, this must be changed to

$$c = \sqrt{\frac{\lambda}{2\pi}\left(g + \frac{4\pi^2 T}{\lambda^2 \rho}\right)}. \qquad (11.20)$$

In practice the ripples are produced by a dipper attached to the prongs of an electrically-driven tuning fork of known frequency n. Then since (see Part IV) the velocity $c = n\lambda$, equation (11.20) may be written

$$T = \frac{\lambda^3 n^2 \rho}{2\pi} - \frac{g\lambda^2\rho}{4\pi^2}. \qquad (11.21)$$

The wave-length λ is found by observing the surface of the liquid with a stroboscopic slit (see Part IV) of the same frequency as the fork. The ripples then appear stationary, and the distance between a given number of them may be found with a travelling microscope.

To show that $c = (g\lambda/2\pi)^{\frac{1}{2}}$ for gravitational waves alone, we assume that, as shown in fig. 12, every particle of liquid in the wave describes a circular path in a vertical plane. Then the circles are described in an anticlockwise direction for waves which are proceeding from left to right. If the time of revolution of a particle in the circle is τ, the horizontal velocity at a crest X is

$$q_1 = c - \frac{2\pi r}{\tau}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.22)$$

while at the trough Y it is

$$q_2 = c + \frac{2\pi r}{\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.23)$$

Now, assuming that the gain in velocity is due to the decrease in gravitational potential energy due to the particle falling a height $h = 2r$, we have

$$q_2^2 = q_1^2 + 2gh = q_1^2 + 4gr. \quad . \quad . \quad . \quad (11.24)$$

Hence, from (11.22), (11.23) and (11.24), by eliminating $(q_2^2 - q_1^2)$,

$$c = \frac{g\tau}{2\pi} = \frac{g\lambda}{2\pi c},$$

or

$$c = \left(\frac{g\lambda}{2\pi}\right)^{\frac{1}{2}}.$$

7. Variations in Surface Tension.

The surface tension is very sensitive to small quantities of impurities on the surface, a state of affairs which is referred to as *contamination*. Solutions have a lower surface tension than the pure solvent, but no simple law of variation with concentration is observed. Rise of temperature results in a fall in the surface tension, and the surface tension disappears a few degrees below the critical temperature (see Part II). The variation with temperature is not a simple one; one of the more useful formulæ which have been proposed to represent the relation is the *Eötvös-Ramsay-Shields formula:*

$$T(Mvx)^{\frac{2}{3}} = K\,(\theta_c - \theta - d), \quad . \quad . \quad . \quad (11.25)$$

where K is a constant, M is the molecular weight of the liquid, v its specific volume, that is the reciprocal of the density, θ_c its critical temperature, θ its actual temperature and d a constant equal numerically to about 7. The quantity x is the *degree of association* of the liquid; this is defined as the molecular weight of the liquid divided by the molecular weight of the unassociated liquid. For example, water contains groups of associated molecules nH_2O where n is 2,

3, &c., as well as ordinary H_2O molecules, and the relative fraction of the groups present with $n = 1, 2, 3$, &c., varies as the temperature changes.

8. Osmosis and Osmotic Pressure.

It was discovered by the Abbé Nollet in 1748 that if certain plant cells are placed in a solution of salt, they shrivel, owing to passage of water outwards through the cell membrane, but that they regain their original size if they are again placed in pure water. This passage of liquid through a cell-membrane in one direction only is termed **osmosis,** and is said to take place through a **semi-permeable membrane.** Some cells, such as red blood corpuscles, when placed in pure water receive

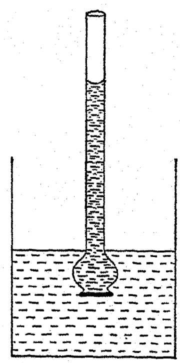


Fig. 13

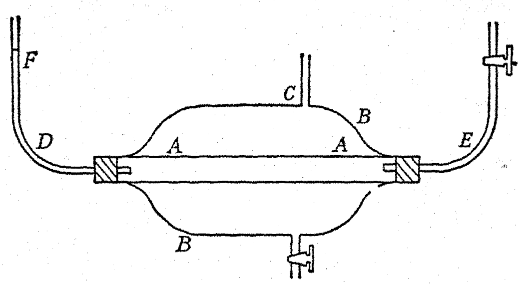


Fig. 14

so much water through their semi-permeable membranes that they burst. This bursting is attributed to **osmotic pressure,** the existence of which may be simply demonstrated with the apparatus shown in fig. 13. A thistle funnel contains some sugar solution and is closed at the bottom with a sheet of parchment which acts as a semi-permeable membrane. The funnel is placed in a beaker of pure water, and as the latter passes through the semi-permeable membrane the height of liquid in the tube of the thistle-funnel continues to rise until the hydrostatic pressure down is just sufficient to balance the osmotic pressure on the membrane. A method of measuring the osmotic pressure is thus provided, but in practice the method is modified so that less strain is placed upon the semi-permeable membrane. The latter is usually made of a film of copper ferrocyanide, deposited in the walls of a porous pot by allowing copper sulphate to diffuse through one side of the pot and potassium ferrocyanide through from the other side. In fig. 14 is shown Berkeley and Hartley's method of measuring osmotic pressure. A horizontal porcelain tube A has a semi-permeable membrane of $CuFe(CN)_6$ deposited near the outer wall. The gun-metal

case B enclosing A is filled with the solution under test by the side-tube C. The brass end-tubes D and E lead respectively to a vertical open graduated glass capillary tube, and to a tap. The solvent is placed in the inner tubes A, D, E, but is prevented from passing into the solution by a hydrostatic pressure applied through C. When the applied hydrostatic pressure is just equal to the osmotic pressure the meniscus at F remains stationary. Osmotic pressures up to 130 atmospheres may be measured in this way.

The fundamental quantitative laws of osmotic pressure are:

(*a*) The osmotic pressure is directly proportional to the concentration of the solution.

(*b*) The osmotic pressure is directly proportional to the absolute temperature.

(*c*) If a gramme-molecule of a non-electrolyte is dissolved in 22·4 litres of solvent an osmotic pressure equal to 76 cm. of mercury is produced at 0° C.

The first two laws may be summed up in the equation

$$p = nkT, \qquad \ldots\ldots \qquad (11.26)$$

where n is the concentration, k is a constant, and T is the absolute temperature. This equation bears a striking resemblance to the gas equation (see Part II), and this, coupled with the fact that law (*c*) is the exact counterpart of Avogadro's hypothesis (see Part II) leads us to attribute osmotic pressure to the same mechanism as gas pressure. We shall see in Part II that the kinetic theory of gases gives us almost a perfect picture of gas pressure arising from the bombardment of gas molecules, so in a similar manner osmotic pressure is attributed to the bombardment of liquid molecules on the semi-permeable membrane. It is assumed that the membrane is permeable to molecules of the solvent but not to those of the solute. Consequently, since all the molecules which impinge on the membrane on the side containing the pure solvent are molecules of solvent, while the latter constitute only a fraction of the total number of bombarding molecules on the other side, there is a net number of solvent molecules passing through the semi-permeable membrane in the direction of solvent to solution. Owing to the complicated nature of the liquid state only partial success has attended this view of osmosis. Also the explanation of the behaviour of electrolytes has required the resources of modern electrical theory. We shall treat osmotic pressure from a different point of view in our Chapter on Thermodynamics (Part II, Heat).

Solutions which exert the same osmotic pressure are termed *isotonic*. Medical injections into the blood-stream must be isotonic with the contents of the blood corpuscles.

9. Diffusion.

Consider the simple experiment of placing a few crystals of copper sulphate at the bottom of a beaker containing water, and then awaiting events. A layer of copper sulphate solution is formed at the bottom of the beaker with a layer of water above it, but after a few months the blue colour is found to have spread uniformly throughout the entire volume of the liquid, indicating that the solution has the same concentration at all points. Now the copper sulphate solution had a much greater density than the pure water, so the question arises as to how the denser liquid can rise against gravity. The phenomenon is referred to as **diffusion** and is explained on the **kinetic theory of liquids.** Since the molecules in the liquid state are endowed with a certain amount of kinetic energy, which is actually proportional to their absolute temperature, this kinetic energy is shared by the molecules of solute present in the solution. The solute molecules therefore wander about in the liquid and owing to their thermal kinetic energy are able to rise against gravitational attraction. The basic **law of diffusion,** due to Fick, states that the *rate of transfer of solute across unit area in the solution is directly proportional to the concentration gradient.* Expressing this mathematically, we have

$$w = -D\frac{dn}{dx}, \quad . \quad . \quad . \quad . \quad . \quad (11.27)$$

where w is the weight of solute crossing unit area per second in the direction of the x-axis and dn/dx is the concentration gradient of the solution at the point in question. The negative sign is introduced because if the solute moves in the direction of increasing x, the concentration must *decrease* in this direction. The rate of transfer also depends on the nature of the solute, and the constant D is called the **coefficient of diffusion** or **diffusivity** for the substance in question.

Diffusion is not confined to the liquid state. Owing to the higher translational velocity of the molecules it is much more rapid in gases. It may be examined by placing two gases in a vertical gas jar, separated by a horizontal glass plate, which may be removed with a minimum of disturbance. For example, a dense gas such as carbon dioxide may fill the lower half of the jar and a lighter gas such as hydrogen may fill the upper half of the jar. If the plate is removed, after some time, it is found that the mixture which is formed by diffusion is of uniform density throughout the vessel. Various means may be employed to study the diffusion of the gas; while chemical analysis is the commonest method, other less disturbing processes, such as the measurement of the refractive index of the gas layers at various heights, may be employed.

Closely related to diffusion is *effusion*. This is the passage of a gas through a small aperture. If a porous pot, which may be regarded as a large number of very small apertures, contains two gases, the rate of effusion will differ and thus a method of separating gases becomes available. This method has been used in separating the isotopes of chlorine, and also heavy hydrogen, from a mixture with ordinary hydrogen (see Part V).

Graham's law of diffusion compares the velocities of diffusion of different gases, and states that the rate at which diffusion (or effusion) takes place through an aperture is inversely proportional to the square root of the density of the gas, that is,

$$v \propto \sqrt{\frac{1}{\rho}}. \qquad (11.28)$$

This equation only holds for constant temperature, the rate of diffusion increasing rapidly as the temperature is raised. A simple apparatus for illustrating gas diffusion is shown in fig. 15. An inverted vertical porous pot containing air has a projecting tube dipping below the surface of liquid contained in a beaker. Over the outside of the porous pot is placed a beaker which has been filled with coal-gas by upward displacement. Bubbles are immediately observed to escape from the end of the tube, the coal-gas diffusing through the porous pot more rapidly than the air diffuses out and so giving rise to an increased pressure inside the pot.

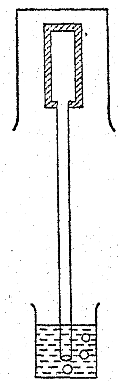


Fig. 15

Even solids show some ability to diffuse over a long period of time. In the experiments of Roberts-Austen, a small lead cylinder was fused to a thin gold plate at one end. After one month the lead cylinder was cut into slices and the quantity of diffused gold found by chemical analysis. In more recent experiments, radioactive lead and ordinary lead have been placed in contact, and the diffusion of the radioactive lead has been measured from the radioactivity which has appeared at various places in the ordinary lead. This is an example of *self-diffusion*, since both substances are lead in their chemical properties, although they have different atomic weights, that is, they are isotopes.

The phenomenon of *diffusion* is closely related with that of *osmosis*, for the latter may be regarded as diffusion in one direction through a semi-permeable membrane. In this connexion the process of **dialysis** is of interest. This is the name given to the passage of some substances termed **crystalloids**, and the non-passage of others termed *colloids*, through certain semi-permeable membranes. For example, mineral acids and salts may be separated from gum by placing a

mixture of the two on one side of a membrane of gold-beater's skin. The former, being crystalloids, pass through the membrane while the latter, being a colloid, remains behind. However, the line of demarcation between crystalloids and colloids is very vague; sodium stearate, for example, acts as a colloid in aqueous solution and as a crystalloid in alcoholic solution.

EXERCISES

1. Define *surface tension* and *surface energy* of a liquid-gas interface.
Describe an *accurate* method for measuring the surface tension of a liquid.

2. How may the interfacial surface tension between two non-miscible liquids be measured?
What variation in surface tension results from (*a*) temperature change, (*b*) presence of impurities?

3. Deduce an expression for the difference of pressure existing on opposite sides of a curved film which is subject to surface tension forces.
Calculate the energy latent in a soap-bubble of radius 5 cm., given that the surface tension of the soap-solution is 35 dynes/cm. [7000 π ergs.]

4. What is meant by the term *angle of contact* as applied to capillarity and how may the angle of contact be measured?
Describe the method of determining surface tension from measurements on stationary drops or bubbles.

5. Enumerate the methods available for measuring the surface tension of liquids and describe one method which is especially suitable when only a small quantity of liquid is available.

6. Distinguish between statical and dynamical methods of measuring surface tension and describe one dynamical method in detail.

7. Give an account of the ripple method for measuring the surface tension of a liquid.

8. If a globe of water of radius 2 cm. suddenly splits into 50 equal globules under isothermal conditions, determine the gain in surface energy which occurs, given that the surface tension of water is 75 dynes/cm. [3225 π ergs.]

9. Show mathematically that a homogeneous sheet of lava which contracts on solidifying will split into hexagonal flagstones (Giant's Causeway) because this geometrical figure is the one consistent with least work being done against surface tension forces.

10. State the main laws governing osmosis for substances which do not dissociate on solution. How may the osmotic pressure of a solution be measured?

11. Write a short essay on the phenomena of diffusion of matter.

CHAPTER XII

Properties of Fluids in Motion

It is true that the processes of osmosis and diffusion are due to motion, but it is only random *molecular motion*, the bulk of the fluid remaining at rest. We shall now discuss the properties of fluids in motion, considering first liquids and then gases.

* 1. Translational Flow of an Ideal Liquid.

We first neglect the relative motion of one layer of liquid over another, which produces *internal friction* or *viscosity*, and consider the behaviour of a frictionless, incompressible liquid. If such a liquid is acted upon by gravitational forces, it has a certain gravitational potential energy, and also a kinetic energy due to its motion. If the pressure in the liquid is p and the liquid is moving vertically in a cylinder the work done on the piston (see fig. 1) if the volume increases by V is pV. If the mass of liquid raised is m and the mean height through which it is raised is h, the work done against the gravitational field is mgh. Finally, if its velocity of movement is v, the kinetic energy imparted to the liquid is $\frac{1}{2}mv^2$. If the system is a conservative one, that is to say, if no energy either enters or leaves it, then by the principle of the conservation of energy

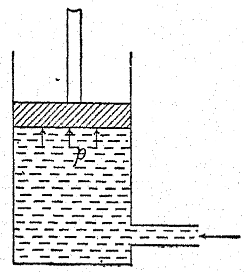


Fig. 1

$$pV + mgh + \tfrac{1}{2}mv^2 = \text{constant}, \quad . \quad . \quad . \quad (12.1)$$

or since the density of the liquid is $\rho = m/V$,

$$p + \rho gh + \tfrac{1}{2}\rho v^2 = \text{constant}. \quad . \quad . \quad . \quad (12.2)$$

This result is known as **Bernoulli's Theorem.** For a liquid which is moving *horizontally*, h is constant and therefore (12.2) becomes

$$p + \tfrac{1}{2}\rho v^2 = \text{constant}. \quad . \quad . \quad . \quad . \quad . \quad (12.3)$$

In equation (12.3) p is termed the *static pressure* and $\frac{1}{2}\rho v^2$ the *dynamic pressure* or *velocity pressure*; the equation therefore states that for an *ideal fluid*, moving horizontally, the sum of the static and dynamic pressures is constant. We may apply equation (12.2) to find the velocity of efflux of liquid escaping from an orifice under the action of gravity. If Π is the atmospheric pressure, and h is the height of the head of liquid above the orifice, then, just outside the orifice, $p = \Pi$, $h = 0$; and, at the upper surface, $p = \Pi$, $v = 0$. Hence

$$\Pi + \tfrac{1}{2}\rho v^2 = \Pi + \rho g h, \quad . \quad . \quad . \quad . \quad (12.4)$$

so that

$$v = (2gh)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.5)$$

* 2. Rotating Ideal Liquid.

We next consider the shape of the surface of liquid contained in a beaker and rotated rapidly about a vertical axis of revolution. As shown in fig. 2, each small element of volume of the liquid is acted upon by the gravitational force downwards and centrifugal force outwards, so for equilibrium we have

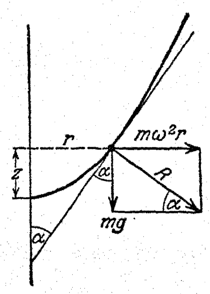


Fig. 2

$$\tan \alpha = \frac{mg}{m\omega^2 r}, \quad . \quad . \quad . \quad (12.6)$$

where m is the mass of the element of liquid, r is the radius of its path, and ω the angular velocity of revolution. The angle is that made with the vertical by the tangent to the liquid surface at the point considered; the resultant force R, which is balanced by the atmospheric pressure, must be perpendicular to the liquid surface for the liquid to rotate steadily. Now $\tan \alpha = dr/dz$, so equation (12.6) may be written

$$dz = \frac{\omega^2}{g}\, r\, dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.7)$$

which gives on integration

$$z = \frac{\omega^2}{2g} r^2 + \text{constant}. \quad . \quad . \quad . \quad . \quad . \quad (12.8)$$

This is the equation to a parabola, so that the surface of the liquid is parabolic, for a uniform speed of rotation.

3. Viscosity.

The behaviour of liquids with translational and rotational motion is approximately described by the considerations of sections (1) and (2), but in practice, internal friction or viscosity must be taken into

account. With reference to fig. 3, it is found that when liquid flows over a fixed surface such as AB, a layer at distance $(x + dx)$ from AB flows with a velocity greater than that at a layer C, distant x from AB. If the difference in the velocities at the two layers is dv, the *velocity gradient* between the two is dv/dx. **Newton's Law of Viscous Flow** states that the viscous force between the two layers is

$$F = \eta A \frac{dv}{dx}, \quad \ldots \ldots \quad (12.9)$$

F acting opposite to the direction of flow and A being the area of either layer. The quantity η is termed the **coefficient of viscosity,** and is characteristic of the liquid in question. From (12.9), η has dimensions $ML^{-1}T^{-1}$. If the velocity gradient is small it is found that the

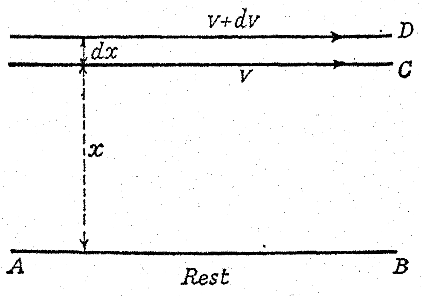


Fig. 3

layers of liquid flow continually parallel to the fixed surface AB. If, however, the velocity gradient is large, local eddies are set up. Newton's law applies only to the former condition, which is that of **stream-line flow,** and not to the latter, which is termed **turbulent motion.**

4. Methods of Measuring η for Liquids.

The methods available for determining the coefficient of viscosity of liquids fall into two groups, the first involving the rate of flow of liquid through a capillary tube, and the second involving the motion of a solid body through a mass of the liquid. In both cases the conditions must be those of stream-line motion and this means that in general the velocity of the liquid or solid must be small.

(*a*) Poiseuille's Method.

An apparatus suitable for the measurement of η by this method is shown in fig. 4. The liquid flows down a horizontal tube of small diameter, the pressure difference between two points being measured by the manometer as shown. Then if Q is the volume of liquid

collected in the measuring cylinder per second, we shall proceed to show that

$$Q = \frac{\pi p a^4}{8 l \eta}, \quad \ldots \ldots \quad (12.10)$$

where l is the length of the tube over which the pressure difference p is acting, a is the mean radius of the tube, and η is the coefficient of viscosity. The liquid is flowing most rapidly at the centre of the tube and is at rest in contact with the walls of the tube. Let the velocities at the inside and outside of an annulus of the liquid between the radii r and $r + dr$ be v and $v + dv$ respectively, as shown in fig. 5. Then

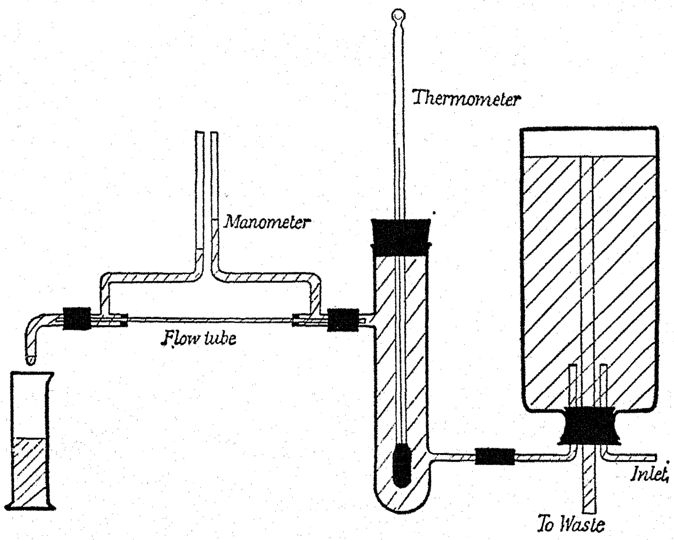


Fig. 4

the velocity gradient across this annulus is dv/dr and by Newton's law of viscous force a tangential stress will be produced, given by $\eta dv/dr$. Since the flow is proceeding with uniform velocity this tangential force exactly balances the accelerating force due to the pressure difference across the ends of the volume of liquid in the cylinder of area πr^2. Hence

$$p \,.\, \pi r^2 = -\eta \,.\, 2\pi r l \,.\, \frac{dv}{dr},$$

or

$$\frac{dv}{dr} = -\frac{rp}{2\eta l}, \quad \ldots \ldots \quad (12.11)$$

so that the velocity gradient is proportional to r. At the wall of the

tube $r = a$ and $v = 0$; therefore, integrating (12.11) from $r = a$ to $r = r$, we have

$$a^2 - r^2 = \frac{4\eta l v}{p}$$

or

$$v = \frac{p}{4\eta l}(a^2 - r^2). \quad \ldots \ldots \quad (12.12)$$

Equation (12.12) shows that $v \propto (a^2 - r^2)$, so that the profile of the advancing liquid is parabolic.

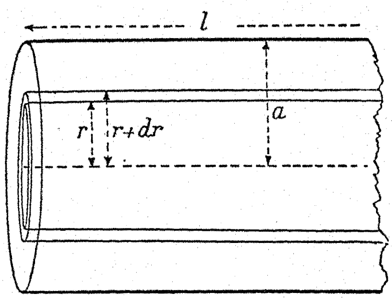


Fig. 5

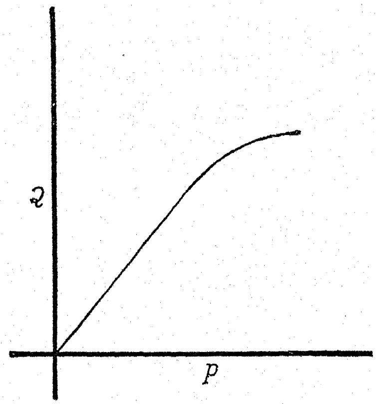


Fig. 6

Now the volume of liquid dQ flowing per second through the annulus of the tube between r and $r + dr$, with velocity v, will be the volume of the annulus multiplied by v, that is

$$dQ = 2\pi r\, dr\, .\, v. \quad \ldots \ldots \quad (12.13)$$

Hence the total volume of liquid flowing through the tube per second is from (12.12) and (12.13),

$$Q = \int_0^a \frac{\pi p}{2\eta l}(a^2 - r^2) r\, dr = \frac{\pi p a^4}{8 l \eta}.$$

We have made no allowance in this deduction of **Poiseuille's equation** for the kinetic energy imparted to the liquid by the pressure difference. An expression for this may be deduced by application of Bernoulli's theorem, discussed on p. 131. However, the correction which results is only approximate, and other corrections of doubtful magnitude are also involved. A more correct formula is

$$\eta = \frac{\pi p a^4}{8Q(l + 1{\cdot}64a)} - \frac{mQ\rho}{8\pi(l + 1{\cdot}64a)}, \qquad (12.14)$$

where ρ is the density of the liquid, and m is a value which varies with the conditions of the experiment, but which is not very different from unity.

The velocity at which turbulent flow sets in is termed the **critical velocity** V_c. It may be shown by dimensional analysis that $V_c = k\eta/\rho a$ where k is a quantity termed the **Reynolds number,** and generally has a value about 1000. The relation between the volume of liquid Q and the pressure p is shown in fig. 6. At low velocities Q is proportional to p, as indicated by Poiseuille's equation, but when the critical velocity is exceeded Q is proportional to $p^{\frac{1}{2}}$. The pressure difference is now occupied in communicating kinetic energy to the eddies as well as in overcoming internal friction.

(*b*) **Rotation Viscometer.**

A rotation viscometer due to Searle is shown in fig. 7(*a*). The inner cylinder a is pivoted about a vertical axis b, and rotates under

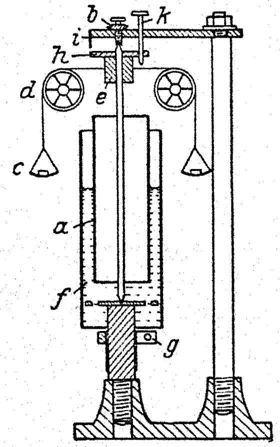


Fig. 7*a*

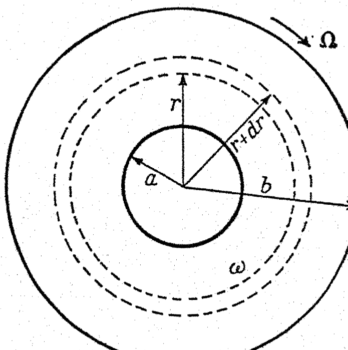


Fig. 7*b*

a couple provided by the weights in the scale-pans c. The couple is transferred by cords passing over the frictionless pulleys d to a drum e. The outer cylinder f can be raised or lowered by rotating the ring g. The speed of rotation is determined with a stop-watch by observing the transits of the point i over the circular scale h. The apparatus is stopped or released by raising or lowering the stop k. In considering the theory of the rotation viscometer, let the angular velocity of rotation increase by $d\omega$ across the annulus between radii r and $r + dr$ as shown in fig. 7(*b*) (in which it is the *outer* cylinder which is shown rotating). Then the velocity gradient across the annulus is $rd\omega/dr$,

so that by Newton's law the viscous force round the annulus is, per unit length of the cylinder,

$$F = 2\pi r\eta \,.\, r\frac{d\omega}{dr}, \quad . \quad . \quad . \quad . \quad . \quad (12.15)$$

and the torque about the central axis is

$$Fr = 2\pi r^3\eta\frac{d\omega}{dr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.16)$$

When the steady state is reached, this torque G, multiplied by the length of the cylinder, l, must equal the steady external couple on the inner cylinder; that is, $Gdr/r^3 = 2\pi\eta l\,d\omega$; integrating, therefore, between the limits

$$r = a, \ \ \omega = \Omega; \ \ r = b, \ \ \omega = 0,$$

we find,

$$\mathrm{G} = 4\pi\eta\Omega\frac{a^2b^2}{(b^2-a^2)}\,l. \quad . \quad . \quad . \quad . \quad (12.17)$$

To eliminate correction for the torque over the base of the cylinder, two lengths are employed. Then if $f(\mathrm{B})$ is the unknown torque over the base

$$G_1 = 4\pi\eta\Omega\frac{a^2b^2}{(b^2-a^2)}\,l_1 + f(\mathrm{B})$$

$$G_2 = 4\pi\eta\Omega\frac{a^2b^2}{(b^2-a^2)}\,l_2 + f(\mathrm{B})$$

so

$$\eta = \frac{(b^2-a^2)(G_1-G_2)}{4\pi a^2b^2\Omega(l_1-l_2)}. \quad . \quad . \quad . \quad . \quad . \quad (12.18)$$

The velocity of rotation must be small so that the conditions are those of stream-line flow.

(*c*) **Stokes's Falling-Body Viscometer.**

This viscometer is extremely simple to use and therefore very convenient. It was shown by Stokes that if a sphere of radius a is moving with uniform velocity v through an infinite, homogeneous, incompressible fluid of coefficient of viscosity η, the retarding force F is given (compare Ex. 3, p. 52) by

$$F = 6\pi\eta av. \quad . \quad . \quad . \quad . \quad (12.19)$$

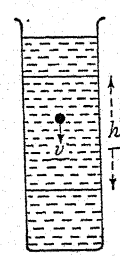


Fig. 8

In practice, the apparatus consists of a vertical glass tube containing the liquid as shown in fig. 8, in which spheres of radius a and mass m are dropped. After a short time the spheres acquire a uniform *terminal*

velocity v, and this may be determined by timing the fall of the spheres over a fixed height h of the tube. Since the accelerating force down $m'g$ must just equal the viscous retarding force when the spheres move with uniform velocity,

$$m'g = 6\pi\eta av, \quad \ldots \ldots \quad (12.20)$$

and hence η may be determined. The quantity m' is the mass of the sphere in the liquid, that is, the buoyancy of the liquid has been taken into consideration. If ρ and σ are the densities of spheres and liquid respectively, equation (12.20) may be written

$$\frac{4}{3}\pi a^3(\rho - \sigma)g = 6\pi\eta av, \text{ or } \eta = \frac{2}{9}\cdot\frac{a^2(\rho - \sigma)g}{v}. \quad (12.21)$$

This equation is of fundamental importance in Millikan's method for determining the electronic charge e, described in Part V.

Equation (12.21) must be divided by a factor $(1 + 2{\cdot}4\, A/R)$ where A and R are the radii of sphere and tube, if accurate valves of η are required. This correction arises from the finite extent of the tube.

*5. Viscosity of Gases.

Newton's law of viscous flow applies to gases as well as to liquids, with the same limitation to stream-line flow. One essential difference is that the high compressibility of gases must also be taken into account. Otherwise, with small modifications the methods of measuring the viscosity of gases are essentially the same as for liquids. The rotating cylinder method may be applied directly; it is usual to rotate the *outer* cylinder and measure the torque on the inner cylinder, which is suspended by a torsion wire. To allow for the compressibility of gases when the coefficient of viscosity is determined by a flow method, either a constant pressure or a constant volume is usually employed.

(*a*) Constant Pressure.

The experimental arrangement due to Schultze is shown in fig. 9. The gas is contained in the spheres a and b and is passed through the capillary flow tube k at constant pressure by raising the mercury c up the scale d. The volume of gas passed is recorded electrically as the mercury passes the metal points inserted at f and g.

To allow for the compressibility of the gas we have, by Poiseuille's equation, for a short length dx of the flow tube across which a pressure difference dp is acting,

$$Q = -\frac{\pi a^4}{8\eta}\cdot\frac{dp}{dx}. \quad \ldots \ldots \quad (12.22)$$

When a steady flow of gas is in progress the *mass* of gas crossing all cross-sections of the tube must be constant. This is an example of the **Equation of Continuity.** It holds for all fluids and for electric current. If it were not true at any point, there would be a continual generation or destruction of fluid at that point. Now the density of the gas is, by Boyle's law, directly proportional to the pressure, so that we have, for any cross-section of the tube,

mass crossing per sec. = constant = vol. per sec. × density.

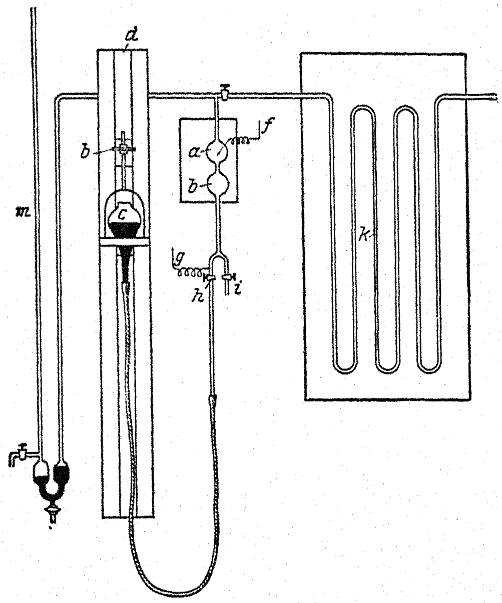


Fig. 9

Thus, if the volume of gas entering the flow tube per second is Q, and its pressure, as registered by the manometer m in fig. 9, is p, then

$$Q\rho \propto Qp. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.23)$$

Also $$p_1Q_1 = pQ = -\frac{\pi a^4}{8\eta}p \cdot \frac{dp}{dx}, \quad . \quad . \quad . \quad (12.24)$$

from equation (12.22). Hence, integrating over the whole length l of the flow tube,

$$\int_0^l p_1Q_1\,dx = -\frac{\pi a^4}{8\eta}\int_{p_1}^{p_2} p\,dp,$$

or $$p_1Q_1 = \frac{(p_1^2 - p_2^2)\pi a^4}{16\eta l}. \quad . \quad . \quad . \quad (12.25)$$

All the quantities in (12.25) are known, except η, so that η may be determined. The pressure p_2 is, of course, atmospheric pressure; correction must be applied for the kinetic energy imparted to the gas, as in the corresponding experiment with liquids.

(*b*) Constant Volume.

In Edward's constant volume method, a diagram of which is shown in fig. 10, the gas is contained in a large bulb B at a pressure p, which is registered by the mercury in the arms of the manometer ab. The tap T_2 is then closed and T_3 opened to atmosphere for a certain time T.

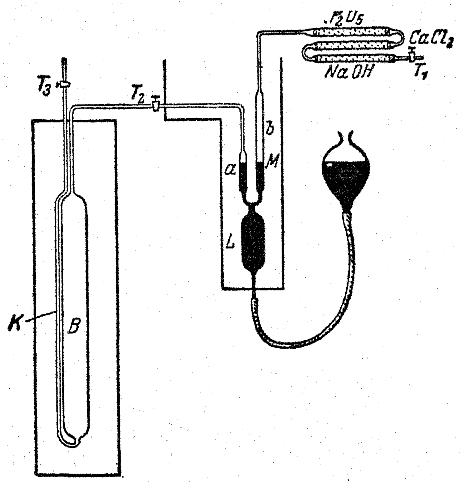


Fig. 10

Finally, T_3 is closed and the new gas pressure p_2 is observed. Let the volume of the bulb be Q; then if Q_1 is the volume of gas entering the capillary flow tube K per second at a time t when the pressure in the apparatus is p_1, for a slow rate of flow we may apply Boyle's law. During the interval dt the volume becomes $(Q + Q_1\,dt)$ while the pressure becomes $(p + dp)$, so

$$pQ = (p + dp)(Q + Q_1\,dt), \quad . \quad . \quad . \quad (12.26)$$

and, neglecting the small quantity $Q_1\,dp\,dt$,

$$pQ_1 = -Q\frac{dp}{dt}.$$

Now by equation (12.25), if p_0 is the atmospheric pressure

$$pQ_1 = \frac{\pi a^4(p^2 - p_0{}^2)}{16\eta l}. \quad . \quad . \quad . \quad . \quad (12.27)$$

Hence

$$\frac{\pi a^4 T}{16\eta l Q} = -\int_{p_1+p_0}^{p_2+p_0} \frac{dp}{p^2 - p_0^2} = \frac{1}{2p_0} \log_e\left(\frac{p_2 + 2p_0}{p_1 + 2p_0} \cdot \frac{p_1}{p_2}\right). \quad (12.28)$$

6. Variations in the Viscosity of Fluids.

The coefficient of viscosity of liquids increases with rise of pressure, and decreases very rapidly with rise of temperature. The viscosity of solutions is irregular and bears no simple relation to the concentration. Singularly little success has attended the attempts to apply the molecular theory to explain the viscosity of liquids.

With gases, the coefficient of viscosity is independent of the pressure at ordinary pressures, a rather unexpected result indicated first theoretically from the kinetic theory of gases and subsequently verified experimentally, by Maxwell, by observation of the damping of an oscillating disk surrounded by gas at different pressures. At low pressures the viscosity varies directly as the pressure, a fact which forms the basis of *viscosity gauges* for measuring low gas pressures (see p. 111). The viscosity of gases, in contrast to that of liquids, increases with rise of temperature. We shall see in Part II that the behaviour of gases is well accounted for by the kinetic theory.

EXERCISES

1. Discuss Bernoulli's theorem concerning the translational flow of an ideal liquid and use it to deduce the velocity of efflux of liquid escaping from an orifice under the pressure of a constant head of water.

2. Show that the surface of an ideal liquid contained in a cylindrical vessel rotating about a vertical axis is a paraboloid of revolution.

A cylindrical vessel of diameter 30 cm. contains water and is rotating with uniform velocity about a vertical axis. If the vessel makes 50 revs/min., find the difference in level between the edges and the centre of the water surface. [3.1 cm.]

3. Liquid stands in a vertical cylinder attached to an automobile which is being accelerated. How will the level of the liquid in the cylinder be affected (*a*) if the cylinder is fixed rigidly to the floor, (*b*) if it is freely suspended from the roof?

4. Explain clearly what is meant by the term **coefficient of viscosity** of a fluid.

Describe one method for finding the coefficient of viscosity of a liquid experimentally.

5. Give an account of the rotation viscometer and indicate what advantages it possesses over Poiseuille's method in finding the coefficient of viscosity of gases.

6. How does the viscosity of fluids vary with temperature? Describe how Stokes's viscometer might be applied to examine this variation.

7. Describe a constant-volume flow method for determining the coefficient of viscosity of a gas.

How does the viscosity of a gas vary with the pressure?

8. If water is flowing down a right-circular cylinder of radius 1 mm. under a pressure gradient of 10 dynes/cm.^{3}, find the velocity of flow at a point 0·5 mm. from the centre of the tube, given that the coefficient of viscosity of water is 0·01 gm./cm.$^{-1}$sec.$^{-1}$. [1·9 cm./sec.]

EXAMPLES

1. Use the method of dimensions to find how the time of oscillation of a simple pendulum varies with (*a*) the mass of the bob, (*b*) the length of the suspension, (*c*) the acceleration due to gravity.

2. Find by the method of dimensions how the volume of viscous liquid flowing per second with streamline motion through a cylindrical tube of circular cross-section depends on (*a*) the pressure gradient, (*b*) the radius of the tube and (*c*) the coefficient of viscosity of the liquid.

3. If a smooth sphere of mass M, moving with uniform velocity, collides with another sphere (initially at rest) of mass m, show that if translational energy and momentum are conserved in the collision, $M/m = \sin(\phi + 2\theta)/\sin\phi$, where ϕ and θ are the angles made by the directions of motion of M and m after collision, with the direction of motion of M before collision.

4. In a sand-blast, spherical particles each of mass 0·1 gm. are sent along a cylindrical tube of cross-section 9·5 cm.^{2}, the end of which is held in contact with a flat plate. If 100 particles impinge on the plate per sec. with a velocity of 5×10^3 cm. per sec., and the coefficient of restitution is 0·9, find the pressure exerted by the particles on the plate.

5. A particle starting from rest slides down a plane inclined at 30° to the horizontal. If the coefficient of friction between particle and plane is $\frac{1}{4}$, show that the ratio of the kinetic energy of the particle at any instant to its loss in potential energy is $(4 - \sqrt{3})/4$.

6. A spring balance hangs from the roof of a stationary lift and supports a weight of 5 lb. The lift starts to descend, whereupon the balance reads 4·5 lb. What is the downward acceleration of the lift, and what would the balance read if the lift fell freely under gravity?

7. A mascot hangs vertically in an automobile which is moving with a uniform velocity of 45 m.p.h. along a straight horizontal road. If the brakes are applied to give a uniform resistance and the mascot is observed to be deflected through an angle of 10°, determine the distance the car travels before coming to rest.

8. A flywheel of mass 100 lb. is mounted on a horizontal axle of diameter 2 in. Determine the radius of gyration of the wheel if a

mass of 1 lb., attached to the axle by a string, descends through 6 ft. from rest in 10 sec.

9. An aeroplane is approaching an air-port at a uniform speed u_1 when another plane leaves the port with a uniform speed u_2. Find the ratio of the distances of the aeroplanes from the port when they are closest to each other if the angle between the lines of motion of the planes is θ.

10. Taking the acceleration due to gravity (observed) as 981 cm./sec.^{2} at latitude 45°, determine its value at latitude 60° if the earth is regarded as a sphere of radius $6{\cdot}4 \times 10^8$ cm.

11. A propeller-driven ice-yacht is made to move in a circle on a rough plane inclined at an angle θ to the horizontal. If the coefficient of side-ways friction is μ, show that the greatest allowable velocity v_1 at the highest point of the circle is related to the greatest allowable velocity v_2 at a point midway between the top and bottom of the circle, by the equation $v_2{}^2/v_1{}^2 = \mu \cos\theta/(\sin\theta + \mu \cos\theta)$.

12. A small pellet is projected from the bottom of a circular hoop of radius 1 ft. with a speed of 12 ft./sec. If it just reaches the top of the hoop, determine the percentage energy loss due to friction.

13. A car is running round a curved track of radius 1000 ft. at a speed of 120 m.p.h. Determine the angle of banking so that there shall be no tendency to side slip.

14. Referring to Ques. 13, if the centre of gravity is 2 ft. from the ground and the wheel base is 5 ft. wide, find the maximum speed at which the car may safely take a corner on a very rough horizontal road, if the radius of the track is 20 ft.

15. A horizontal platform executes simple harmonic motion in a vertical plane, the total vertical movement being 10 cm. Find the shortest period permissible if objects resting on the platform are to remain in contact with it throughout the motion.

16. Show that if a short straight frictionless tunnel is driven through the earth from one point on the surface to another, then assuming the earth to be a sphere of radius 4000 miles, an object will travel along the tunnel under gravitational forces and will again reach the surface in about 42 min.

17. A uniform open U-tube contains very mobile liquid to a height of 50 cm. in each limb. If the liquid in one limb is depressed and then released, find the period of oscillation of the liquid in the tube.

18. What is the length of the equivalent simple pendulum for a pendulum which consists of a sphere of radius 15 cm., suspended by a light string 120 cm. long?

19. Show that in a reversible pendulum which has been nearly adjusted, except when the centre of gravity is very close to the point midway between the two knife-edges, g is given by $g = 8\pi^2 l/(T_1^2 + T_2^2)$, where l is the distance between the two knife-edges and T_1 and T_2 are the nearly equal times of oscillation.

20. A circular hoop is suspended by any number of vertical strings of length l, uniformly spaced around the hoop. Show that the period of a small rotational oscillation about the centre is that of a simple pendulum of length l.

21. A wire that breaks under a tension greater than 3×10^9 dynes carries a smooth ring which supports a weight of 4 kgm. If the wire is pulled in a horizontal direction, determine the angle the wire makes with the horizontal at the instant of fracture.

22. A given vertical light spiral spring is observed to stretch 1 cm. when loaded with a weight of 50 gm. Find the velocity with which a 1000 gm. weight is travelling if it is attached to the end of the spring and has then fallen through 10 cm.

23. In Ques. 22 determine the maximum extension of the spring and the period of oscillation of the 1000 gm. mass.

24. Given that Young's modulus for brass is 10^{12} dynes/cm.^{2}, find the energy stored up in a stretched brass wire of area of cross-section 1 mm.^{2} and initial length 1 m., when it is loaded with 2000 gm.

25. A load of 2 kgm. is observed to stretch a given wire through 1 mm. Determine the work done in stretching the wire through 5 mm. and the load required to do it.

26. Two vertical wires of copper and brass are initially of the same length, but the former has twice the diameter of the latter. If Young's modulus for the two materials has the ratio 12 : 11, find the ratio of the extensions when they are stretched by the same load.

27. If the bulk modulus for the compression of a liquid is k and its thermal coefficient of cubical expansion is γ, find the pressure that must be applied to the liquid to prevent it expanding when its temperature is raised t°.

28. A rectangular bar of length 20 cm. is supported at its extremities by a bifilar suspension of two vertical strings of length 100 cm. If the mass of the bar is 800 gm., determine the restoring couple acting upon it when it is twisted through 0·001 radian.

29. Find the time of oscillation of the bar in Ques. 28 if its breadth may be neglected in comparison with its length.

30. Determine the greatest weight that may be lifted by a person with the aid of a jack, given that the person can exert a maximum

force of 80 lb. on a 3-ft. lever attached to the screw which has 10 threads to the inch.

31. In an arrangement of pulleys, 3 are attached to an upper block which is fixed, the remaining 2 being attached to a lower movable block. If a man of weight 180 lb. stands on the lower block which weighs 20 lb., find the force which he must exert to raise himself with the aid of a rope attached to the lower block and passing round all the pulleys. How much work has been done when the free end of the rope has descended through 50 ft.?

32. If the gravitational constant $G = 6{\cdot}66 \times 10^{-8}$ c.g.s. units and the acceleration due to gravity is 32 ft./sec.^{2}, find the mean density of the earth, taking the latter as a sphere of radius 4000 miles.

33. With a given Nicholson's hydrometer it is found necessary to add a weight of 50 gm. to the scale-pan in order to sink it to the required mark. On placing an insoluble solid in the top pan, the weight required is only 25 gm., whereas this has to be increased to 30 gm. when the solid is placed in the lower scale-pan immersed in the liquid. Find the volume and the density of the solid.

34. Two vertical cylindrical tanks have areas of cross-section 10 cm.^{2} and 5 cm.^{2} respectively, and are joined at the base by a U-tube of area of cross-section 1 cm.^{2}. Liquid of specific gravity 2 is then poured into the U-tube so as to half-fill it and water (immiscible with the liquid) is then added to both tanks until it stands well above the bottom of each tank. If the gas pressure above the two tanks is initially the same and that above the larger is then increased by an amount equal to 2 cm. of water, what is the change in the difference of height between the two columns of liquid in the U-tube?

35. A uniform wooden pole of length 10 ft., and specific gravity 0·6 lies with one end on the bank of a lake and the other end dipping into and supported by the water, the level of which is only a few inches below the edge of the lake. Find the immersed length of pole and show that within certain limits it is independent of the depth of the water surface below the bank.

36. Determine the greatest height h to which water may reach in a reservoir with rectangular ends of breadth a, if the end collapses when a couple greater than G acts about the bottom edge of the end of the reservoir.

37. A quarter of a circular cylindrical surface of radius r and length l is immersed in water with one edge of length l in the surface and the axis of the cylinder vertically below it. Show that the resultant force on the surface is about $0{\cdot}54\ gr^2l$, and is inclined to the horizontal at about 23°.

38. A thin flat plate in the shape of a triangle of vertical height h is immersed in a liquid in a vertical plane with its base horizontal and at a distance $2h$ below the surface, the vertex being above the base. Show that the centre of pressure of the triangle is at a depth $17h/10$.

39. A cylindrical vessel contains water and a floating body. Show that provided that the body has constant volume, the water level in the cylinder is independent of the area of cross-section of the body.

40. In Ques. 39 if the height of the water in the cylinder before the introduction of the body is 30 cm. and the body weighs 20 gm., find the force required to submerge it, given that when it floats in the water the level rises to 34 cm. whereas when it is completely submerged the level reaches 36 cm.

41. Taking the density of air at a temperature of 0° C. and a pressure of 760 mm. of mercury to be $1{\cdot}29 \times 10^{-3}$ gm./c.c., find to what depth a hollow metal sphere must be sunk in sea-water of constant density 1·03, before the air will start to leak out through a crack in the bottom of the sphere. The temperature of the sea-water is to be taken as 0° C. and the density of mercury as 13·6 gm./c.c.

42. A vertical closed cylinder is floating in equilibrium and is immersed to a depth of 10 cm. Determine the period of vertical oscillation of the cylinder if it is depressed a further small distance into the liquid and is then released.

43. Determine the osmotic pressure of 1 per cent solution of cane sugar at 100° C.

44. A circular rubber band has an unstretched diameter of 5 cm. and requires a tension of $910/\pi$ dynes to increase its length by 1 cm. It is placed on a soap-film and the film inside the ring is then broken. If the new diameter of the ring is observed to be 5·2 cm., determine the surface tension of the soap-film.

45. A drop of liquid of surface-tension T is placed between two flat plates. Show that if the liquid is spread out into a circular patch of area A, the force required to separate the plates when the separation is d is $2\mathrm{TA}/d$.

46. If the surface-tension of some soap solution is 35 dynes/cm., find the excess pressure inside a soap bubble of diameter 10 cm. Find also the work done in blowing this bubble.

47. A hollow vertical cylindrical tube of weight 20 gm. has a radius of 1 cm. and is closed at the base. Determine the length of the tube immersed in water if the surface tension of the latter is 75 dynes/cm. and the tube is floating in equilibrium.

48. Determine the difference in level between the surfaces of mercury in a U-tube having limbs of diameter 1 mm. and 0·5 mm.

respectively, given that the surface tension of mercury is 550 dynes/cm., its density is 13·6 gm./c.c., and the angle of contact is 140°.

49. Two tubes A and B of lengths 100 cm. and 50 cm. have radii 0·1 and 0·2 mm. respectively. If liquid is passing through the two tubes, entering at A at a pressure of 80 cm. of mercury and leaving B at a pressure of 76 cm. of mercury, find the pressure at the junction of A and B.

50. Bubbles of the same diameter are blown at the end of a long narrow tube using successively two liquids A and B. If the times taken for the bubbles to collapse are 30 sec. and 20 sec. respectively, what is the ratio of the surface tensions of A and B?

ANSWERS AND HINTS FOR SOLUTION

1. $t = f(m, l, g)$; dimensionally $t = \mathrm{T}$, $m = \mathrm{M}$, $l = \mathrm{L}$, $g = \mathrm{LT}^{-2}$. Suppose $t \propto m^{\alpha} l^{\beta} g^{\gamma}$, so dimensionally $\mathrm{T} = \mathrm{M}^{\alpha}\mathrm{L}^{\beta}\mathrm{L}^{\gamma}\mathrm{T}^{-2\gamma}$; equating indices, $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = -\frac{1}{2}$; hence $t \propto (l/g)^{\frac{1}{2}}$.

2. $\mathrm{V}/t = f(p/l, r, \eta) \propto (p/l)^{\alpha} . r^{\beta} . \eta^{\gamma}$; dimensionally η is given by: viscous force = area × velocity gradient × η; whence $\eta = \mathrm{ML}^{-1}\mathrm{T}^{-1}$; also $r = \mathrm{L}$ and $p/l = \mathrm{ML}^{-2}\mathrm{T}^{-2}$; $\mathrm{V}/t \propto p/l \,.\, r^4 \,.\, 1/\eta$.

3. If U and V are the velocities of M before and after collision and v is the velocity given to m, then (a) conservation of momentum along the initial direction of motion gives $\mathrm{MU} = \mathrm{MV}\cos\phi + mv\cos\theta$, ($b$) conservation of momentum perpendicular to previous direction gives $\mathrm{MV}\sin\phi = mv\sin\theta$, and ($c$) conservation of translational energy gives $\mathrm{MU}^2 = \mathrm{MV}^2 + mv^2$. Eliminate U, V and v.

4. Momentum brought up per sec. $= 100 \times 0{\cdot}1 \times 5 \times 10^3 = 5 \times 10^4$ and momentum taken away is $-0{\cdot}9 \times 5 \times 10^4$; total change in momentum per sec. is $9{\cdot}5 \times 10^4$, and by Newton's second law of motion, this is the force. Hence pressure = force/area = 10^4 dynes/cm.^{2}.

5. Loss in potential energy of particle after descending vertical distance h is mgh. Force on particle down plane is $mg\sin 30° - \frac{1}{4}mg\cos 30°$; gain in kinetic energy after travelling distance $h\,\mathrm{cosec}\,30°$ is therefore $(mg\sin 30° - \frac{1}{4}mg\cos 30°) \,.\, h\,\mathrm{cosec}\,30°$.

6. If downward acceleration of lift is f, considering forces on weight, $5g - 4{\cdot}5g = 5 \,.\, f$; $f = 3{\cdot}2$ ft./sec.^{2}; zero.

7. Considering forces on deflected mascot, if horizontal deceleration of car is f, $f/g = \tan 10°$; also $v^2 - u^2 = 2fs$; or $s = 386$ ft.

8. Moment of inertia × angular acceleration = applied couple. I × ang. acc. $= 1 \times 32 \times 1/12$, since acceleration is small; also $s = ut + \frac{1}{2}ft^2$ or $f = 12/100$ ft./sec.^{2}. Hence ang. acc. $= 12 \times 12/100 = 144/100$; therefore $\mathrm{I} = 100/54$ and $k^2 = \mathrm{I}/\mathrm{M} = 1/54$ ft.^{2}.

9. Let distance of first plane from port be d at instant second plane starts from port, and let least separation of planes occur at subsequent time t. Then $s^2 = (d - u_1t)^2 + u_2^2t^2 - 2u_2t(d - u_1t)\cos\theta$; for s to be a minimum $ds/dt = 0$; hence $t = d(u_1 + u_2\cos\theta)/(u_1^2 + u_2^2 + 2u_1u_2\cos\theta)$, so required ratio $(d - u_1t)/u_2t = (u_2 + u_1\cos\theta)/(u_1 + u_2\cos\theta)$.

10. Angular velocity of rotation of earth is $2\pi/24 \times 60 \times 60$ radians per sec. Hence $r\omega^2 = 3{\cdot}4$ cm./sec.^{2}. Observed acceleration g is resultant of g_0 for a non-rotating earth and centrifugal acceleration due to rotation.

Hence $g^2 = g_0^2 + r^2\omega^2 \cos^2\lambda - 2g_0 r\omega^2 \cos\lambda$ or approximately $g = g_0 - r\omega^2 \cos^2\lambda$, neglecting $r^2\omega^4 \cos^2\lambda$ and expanding the R.H.S. by the binomial theorem. Hence $g_0 = 982{\cdot}7$ and $g_{60°} = 981{\cdot}85$ cm./sec.

11. At highest point, equation of motion in a circle gives $v_1^2/r = g(\sin\theta + \mu \cos\theta)$; at mid-point $v_2^2/r = \mu g \cos\theta$.

12. Kinetic energy of pellet is $m \times 144/2 \times 32 = 9m/4$ ft.-lb. Gain in potential energy is $2m$ ft.-lb. Hence loss in energy due to friction is 11·1 per cent.

13. Equation of equilibrium, neglecting friction, is

$$v^2 \cos\theta/r = g \sin\theta; \;\; 44° \text{ approx.}$$

14. Taking moments about the line of contact of the off-side wheels and the road, $2v^2/r = 2{\cdot}5g$; 28·3 ft./sec.

15. Objects will leave platform if vertical acceleration of latter $> g$. Equation of motion of platform is $x = a \cos\omega t$; hence maximum value of acceleration is $d^2x/dt^2 = -a\omega^2$. Hence $\omega = (g/a)^{\frac{1}{2}}$ and $t = 2\pi/\omega = 0{\cdot}45$ sec.

16. Component of gravitational force along tunnel at any instant is mgx/r where x is distance along tunnel measured from its deepest point; hence motion is simple harmonic, and period is $t = 2\pi(r/g)^{\frac{1}{2}}$, and time taken is $t/2 = 42$ min.

17. If depression from equilibrium level is x, restoring force is $2x \,.\, \mathrm{A} \,.\, \rho \,.\, g$, where A is area of cross-section and ρ is density. Motion is S.H., equation being $2x\mathrm{A}\rho g + 100\mathrm{A}\rho \times$ acceleration $= 0$. Period $t = 2\pi(50/g)^{\frac{1}{2}} = 1{\cdot}42$ sec.

18. Equation of motion about point of suspension is $m[k^2 + (l + a)^2] \times d^2\theta/dt^2 + mg(l + a)\theta = 0$; length of equivalent simple pendulum is therefore

$$[2a^2/5 + (l + a)^2]/(l + a) = 135{\cdot}67 \text{ cm.}$$

19. If k^2 is radius of gyration of pendulum about axis through C.G., then $\mathrm{T}_1 = 2\pi[(k^2 + h_1^2)/gh_1]^{\frac{1}{2}}$ and $\mathrm{T}_2 = 2\pi[(k^2 + h_2^2)/gh_2]^{\frac{1}{2}}$, where h_1 and h_2 are the distances of the points of suspension from the C.G. Eliminate k^2; hence $4\pi^2/g = (h_1\mathrm{T}_1^2 - h_2\mathrm{T}_2^2)/(h_1^2 - h_2^2) = \frac{1}{2}(\mathrm{T}_1^2 + \mathrm{T}_2^2)/(h_1 + h_2) + \frac{1}{2}(\mathrm{T}_1^2 - \mathrm{T}_2^2)/(h_1 - h_2)$ by partial fractions. Since $\mathrm{T}_1 \sim \mathrm{T}_2$ but $h_1 \neq h_2$, the second term on the R.H.S. is negligible.

20. Let tension in each of n strings $= \mathrm{T} = \mathrm{M}g/n$, where M is mass of hoop. Restoring force is $n\mathrm{T}\theta$ where strings are inclined at θ to vertical, and couple about centre is $n\mathrm{T}\theta a = n\mathrm{T}a^2\phi/l$, where l is length of string, a is radius of hoop, and ϕ is rotation of hoop in horizontal plane. Equation of motion is $\mathrm{M}a^2g\phi/l + \mathrm{M}a^2 \times$ ang. acc. $= 0$; hence $t = 2\pi(l/g)^{\frac{1}{2}}$.

21. Resolving forces vertically, if limiting tension in wire is T, $2\mathrm{T}\sin\alpha = \mathrm{M}g$; 2·25 min.

22. Kinetic energy of mass = loss in gravitational potential energy less potential energy stored in spring, that is $\frac{1}{2}mv^2 = mga$ − spring energy. To calculate spring energy, let F = tension when extension is x; then

work done in stretching a further distance dx is F dx, and total work done in stretching spring through distance a is $\int_0^a \mathrm{F}\,dx$. Now $\mathrm{F} = kx$, where k is constant; hence potential energy in spring is $\frac{1}{2}ka^2$. Actually $k = 50 \times 981$, $a = 10$ and $m = 1000$, hence $v = 121{\cdot}4$ cm./sec.

23. Maximum extension occurs when kinetic energy is zero; hence $mgh = \frac{1}{2}kh^2$; $h = 40$ cm.; $t = 2\pi(m/k)^{\frac{1}{2}} = 0{\cdot}9$ sec.

24. Energy stored in wire is $\frac{1}{2}q\mathrm{A}a^2/l$, where a is extension, l is original length, A is area of cross-section, and q is Young's modulus. Alternatively, $a = \mathrm{P}l/q\mathrm{A}$; hence $\mathrm{E} = \frac{1}{2}\mathrm{P}^2l/q\mathrm{A}$, where P is the stretching force; $1{\cdot}9 \times 10^4$ ergs.

25. Since 2 kgm. weight stretches wire through 1 mm., a weight of 10 kgm. will be required to stretch the wire through 5 mm. Work done will be loss in potential energy of weight or $4{\cdot}9 \times 10^6$ ergs.

26. $\mathrm{P}l/\mathrm{A}_1\alpha_1 = q_1$, $\mathrm{P}l/\mathrm{A}_2\alpha_2 = q_2$. Hence $\alpha_1/\alpha_2 = 11/48$.

27. From definitions, $\gamma = (v_t - v_0)/v_0t$, $\mathrm{P} = k(v_t - v_0)/v_0 = k\gamma t$.

28. If θ is angle at which strings are inclined to the vertical, $\theta = 10 \times 10^{-3}/100 = 10^{-4}$ rads. Resolving forces vertically, $2\mathrm{T}\cos\theta = 800$, or $2\mathrm{T} = 800$ approx.; horizontally, couple about vertical axis through centre of rod is $2\mathrm{T}\theta \,.\, 10 = 0{\cdot}8$ gm. cm.

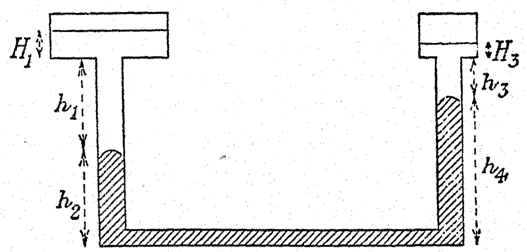


Fig. 1

29. Equation of motion about vertical axis through centre of rod is $\mathrm{I} \times$ ang. acc. $+ 2\mathrm{T}r\theta = 0$, or $\mathrm{I} \times$ ang. acc. $+ \mathrm{M}gr^2\phi/l = 0$, whence time of oscillation $t = 2\pi(\mathrm{I}l/\mathrm{M}gr^2)^{\frac{1}{2}}$. For rectangular bar of small width $\mathrm{I} = \mathrm{M}r^2/3$, hence $t = 1{\cdot}16$ sec.

30. Let weight W be lifted 1 in.; then distance through which person exerts 80 lb. force is $2 \,.\, \pi \,.\, 3 \,.\, 10$ ft., and work done is $4800\,\pi$ ft.-lb. Hence $\mathrm{W} \times 1/12 = 4800\pi$; $180\pi/7$ tons.

31. $6\mathrm{T} = 200$, so $\mathrm{T} = \dfrac{100}{3}$ lb.; $\dfrac{5000}{3}$ ft.-lb.

32. $mg = \mathrm{G}m\mathrm{M}/\mathrm{R}^2$ or $\rho = 3g/4\pi\mathrm{GR}$; $5{\cdot}43$ gm./c.c.

33. Weight of body in air $50 - 25 = 25$ gm. If V = vol. of body, then for equilibrium when body is in lower pan $50 + \mathrm{V} = 30 + 25$, so $\mathrm{V} = 5$ c.c. Since $\mathrm{V}\rho = 25$, $\rho = 5$ gm./c.c.

34. Let initial heights be as shown in fig. 1, and let final heights have primed values. For equilibrium, if initial gas pressure is p, $p + h_1 + \mathrm{H}_1 + h_2\rho = p + h_3 + \mathrm{H}_3 + h_4\rho$, or $\Delta h \,.\, \rho = (h_1 + \mathrm{H}_1) - (h_3 + \mathrm{H}_3)$, where Δh is initial difference between levels of liquid. Finally $2 + (h_1' + \mathrm{H}_1') - (h_3' + \mathrm{H}_3') = \Delta h' \,.\, \rho$. Equations of continuity give, since liquid is incompressible, $(h_1 - h_1')1 = (\mathrm{H}_1' - \mathrm{H}_1)10$, $(h_3' - h_3)1 = (\mathrm{H}_3 - \mathrm{H}_3')5$; also $(h_2 - h_2') = (h_4' - h_4) = \Delta\mathrm{H}/2 = (h_1' - h_1) = (h_3 - h_3')$ where $\Delta\mathrm{H} = \Delta h' - \Delta h$. Putting $\rho = 2$, $\Delta\mathrm{H} = 40/23$ cm.

35. Taking moments about point of contact of pole and bank, if x is length immersed, $10 \times 0{\cdot}6 \times 5 = x(10 - x/2)$ or $x^2 - 20x + 60 = 0$; 3·68 ft.

36. Pressure on side when depth is h is $ha \,.\, \frac{1}{2}\rho g h$; this acts effectively at centre of pressure $h/3$ from base. Hence limiting height is given by $h^3 a\rho g/6 = \mathrm{G}$ or $h = (6\mathrm{G}/ag)^{1/3}$ for water.

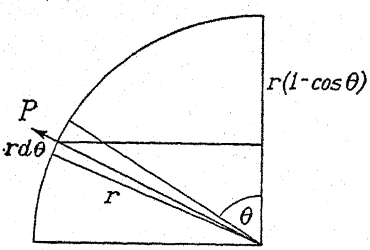


Fig. 2

37. Referring to fig. 2, force on an elementary strip of width $r\,d\theta$ and depth $r(1 - \cos\theta)$ is $\rho g r^2 l(1 - \cos\theta)\,d\theta$ normal to surface. Hence total vertical component of force $\mathrm{V} = \int_0^{\pi/2} \rho g r^2 l(1 - \cos\theta)\, \cos\theta\, d\theta$, and total horizontal component $\mathrm{H} = \int_0^{\pi/2} \rho g r^2 l(1 - \cos\theta)\, \sin\theta\, d\theta$.

38. Divide the triangle into elementary strips of width dx measuring x from the vertex; force on strip parallel to base is $2x\Delta \,.\, dx \,.\, \rho g(h + x)/h^2$, where Δ is area of triangle. Hence centre of pressure is distance x from vertex where

$$x = \int_0^h x^2(h + x)\,dx \Big/ \int_0^h x(h + x)\,dx = 7h/10.$$

39. If density of body is ρ and volume is V, then if equilibrium is reached when immersed volume is v, $\mathrm{V}\rho = v$, which is independent of area of cross-section.

40. Let immersed volume of floating body $= v$. For equilibrium, $v = 20$. If area of cross-section of cylindrical vessel is A, $(34 - 30)\mathrm{A} = v$ or $\mathrm{A} = 5$ cm.^{2}. When completely submerged $(36 - 30)\mathrm{A} = \mathrm{V}$ or $\mathrm{V} = 30$ c.c.; hence $\rho = 2/3$. Additional upthrust on complete immersion is $(\mathrm{V} - v) = 10$ gm.

41. $p/\rho =$ constant; air leaks out when its density exceeds 1·03; 8000 metres.

42. Restoring force for further depression x is $xA\rho g$; hence equation of motion is $10A\rho \times$ acc. $+ xA\rho g = 0$, where A is area of cross-section of cylinder and ρ is density of liquid; $t = 2\pi(10/g)^{\frac{1}{2}} = 0{\cdot}63$ sec.

43. $C_{12}H_{22}O_{11}$ has molecular weight in gm. = 342, and this dissolved in $22{\cdot}4 \times 10^3$ c.c. of water exerts an osmotic pressure of 76 cm. of mercury at 0° C. Hence a 1 per cent solution at 100° C. exerts an osmotic pressure $76 \times 22{\cdot}4 \times 10^3 \times 373/342 \times 100 \times 273 = 68$ cm. of mercury.

44. To find relation between surface tension S and tension T in band, let radius be r and apply principle of virtual work. Then we have $2\pi\, dr\, .\, T = 2\, .\, 2\pi r S\, .\, dr$, or $T = 2rS$. Also $T/0{\cdot}2\pi = 910/\pi$ or $T = 182$ dynes. Hence $S = 182/5{\cdot}2 = 35$ dynes/cm.

45. Consider shape of film at edge; it has two radii of curvature in directions at right angles, $R_1 \approx d/2$ and R_2, where $\pi R_2^2 = A$; hence excess pressure acting over film is $p = T(1/R_1 \pm 1/R_2) = 2T/d$, since $R_2 \gg R_1$. Hence total force urging plates together is $pA = 2TA/d$.

46. $p = 4T/R = 28$ dynes/cm.². Work done in increasing radius from r to $r + dr$ is $p\,dV = 4T\, .\, 4\pi r^2 dr/r$, and total work done in blowing bubble of radius 5 cm. is $16\pi T\int_0^5 r\,dr = 7000\pi$ ergs.

47. If immersed length is l, equilibrium occurs when $\pi\, .\, 1\, .\, l = 2\pi\, .\, 1\, .\, 75/981 + 20$; hence $l = 6{\cdot}52$ cm.

48. Equation of equilibrium for each arm is $2\pi r T \cos\psi = \pi r^2 H\rho g + \pi r^3 \rho g(\sec\psi + \frac{2}{3}\tan^3\psi - \frac{2}{3}\sec^3\psi)$, where ψ is the angle of contact, and where first term on R.H.S. is due to volume of cylinder of supported liquid and second term is due to volume of meniscus; 1·342 cm.

49. Applying continuity equation, volume per sec. passing through tubes is $V = \pi(p_1 - p)r_1^4/8l_1\eta = \pi(p - p_2)r_2^4/8l_2\eta$; 76·12 cm. of mercury.

50. If Poiseuille's equation holds, $V \propto$ excess pressure $\propto T/R$, but time taken is inversely proportional to V; 2 : 3.

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